Control Theory (035188)
lecture no. 7

Leonid Mirkin

Faculty of Mechanical Engineering
Technion — IIT

Outline

Controllability: definition and criterion

Consider

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

This system (or the pair (A, B)) is said to be

- controllable if for every x_0 and x_1, $\exists\, t_1$ and $u(t) : [0, t_1] \to \mathbb{R}$ such that $x(t_1) = x_1$.

Matrix

$$M_c := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times n}$$

called the controllability matrix.

Theorem
(A, B) is controllable if and only if $\det M_c \neq 0$.

Preliminary simplifying observation

Without loss of generality we can take $x_0 = 0$. Indeed,

$$x(t_1) = e^{At_1} x_0 + \int_0^{t_1} e^{A(t_1 - s)} Bu(s) ds$$

implies

$$\tilde{x}(t_1) := x(t_1) - e^{At_1} x_0 = \int_0^{t_1} e^{A(t_1 - s)} Bu(s) ds.$$

Thus, moving $x(t)$ from $x(0) = x_0$ to $x(t_1) = x_1$ is equivalent to moving it from $x(0) = 0$ to $x(t_1) = \tilde{x}(t_1)$. If every $\tilde{x}(t_1)$ is reachable from $x(0) = 0$, then every $x_1 = \tilde{x}(t_1) + e^{At_1} x_0$ is reachable from $x(0) = x_0$.

Remark Alternatively, we may take $x_1 = 0$, based on the relation

$$0 = e^{At_1}(x_0 - e^{-At_1} x(t_1)) + \int_0^{t_1} e^{A(t_1 - s)} Bu(s) ds,$$

which is possible because e^{At_1} is always invertible.

Outline of the proof

Remember, $e^{At} = \sum_{i=0}^{n-1} g_i(t)A^i$. Taking $x(0) = 0$,

$$\begin{aligned} x(t_1) &= \int_0^{t_1} e^{A(t_1-t)}Bu(t)dt = \int_0^{t_1}\left(\sum_{i=0}^{n-1} A^i g_i(t_1-t)\right)Bu(t)dt \\ &= \sum_{i=0}^{n-1} A^i B \int_0^{t_1} g_i(t_1-t)u(t)dt = \sum_{i=0}^{n-1} A^i B\, \eta_i \\ &= M_c\eta, \end{aligned}$$

where $\eta_i := \displaystyle\int_0^{t_1} g_i(t_1-t)u(t)dt$. We have:

1. If $\det M_c = 0$, then $\exists x_1$ such that $x_1 = M_c\eta$ is not solvable in η, hence this x_1 is not reachable by $u(t)$.
2. If $\det M_c \neq 0$, then any x_1 is reachable with $\eta = M_c^{-1}x_1$. It can then be shown that n equations $\eta_i = \int_0^{t_1} g_i(t_1-t)u(t)dt$ are always solvable in $u(t)$ (because of linear independence of $g_i(t)$). □

Controllability and similarity

If $\hat{A} = TAT^{-1}$ and $\hat{B} = TB$ for some nonsingular T, then

$$\begin{aligned} \hat{M}_c &:= \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \cdots & \hat{A}^{n-1}\hat{B} \end{bmatrix} = T\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \\ &= TM_c, \end{aligned}$$

i.e.

- controllability is not affected by similarity transformations.

Controllability: some other tests

Theorem

The following statements are equivalent:

1. (A, B) *is controllable;*
2. $\det M_c \neq 0$;
3. $\det W_c(t) \neq 0$ *for all* $t > 0$, *where* $W_c(t) := \displaystyle\int_0^t e^{As}BB'e^{A's}ds \in \mathbb{R}^{n\times n}$;
4. $\begin{bmatrix} A - \lambda I & B \end{bmatrix} \in \mathbb{C}^{n\times n+1}$ *has full row rank* $\forall\lambda \in \mathbb{C}$ *(PBH test);*
5. $\tilde{\eta}'B \neq 0$ *for every left eigenvector* $\tilde{\eta}$ *of* A;
6. *eigenvalues of* $A + BK$ *can be freely assigned by* $K \in \mathbb{R}^{1\times n}$.

Matrix $W_c(t)$ and control law for $0 = x_0 \to x_1$

Consider

$$u(t) = u_{\min}(t) := B'e^{A'(t_1-t)}[W_c(t_1)]^{-1}x_1.$$

Then

$$\begin{aligned} x(t_1) = \int_0^{t_1} e^{A(t_1-s)}Bu(s)ds &= \int_0^{t_1} e^{A(t_1-s)}BB'e^{A'(t_1-s)}[W_c(t_1)]^{-1}x_1 ds \\ &= \int_0^{t_1} e^{As}BB'e^{A's}ds\,[W_c(t_1)]^{-1}x_1 = x_1. \end{aligned}$$

In fact, $u_{\min}$ has

- minimal energy, $E_u := \displaystyle\int_0^{t_1} u'(t)u(t)dt$,

among all control laws bringing $x(t)$ from 0 to x_1.

Minimum energy proof

If $u(t) = u_{\min}(t) + u_\delta(t)$, then, by linearity,

$$x(t_1) = x_1 + \int_0^{t_1} e^{A(t_1-s)} B u_\delta(s) ds.$$

Hence, $x(t_1) = x_1$ iff $u_\delta(t)$ satisfies

$$\int_0^{t_1} e^{A(t_1-s)} B u_\delta(s) ds = 0.$$

Now,

$$\begin{aligned} E_u &= \int_0^{t_1} (u_{\min}(t) + u_\delta(t))'(u_{\min}(t) + u_\delta(t)) dt \\ &= E_{u_{\min}} + E_{u_\delta} + 2x_1'[W_c(t_1)]^{-1} \int_0^{t_1} e^{A(t_1-t)} B u_\delta(t) dt = E_{u_{\min}} + E_{u_\delta} \end{aligned}$$

(remember Pythagoras). As $E_u \geq 0$, the minimum is attained by $u_\delta(t) \equiv 0$.

Uncontrollable modes

If (A, B) not controllable, PBH test doesn't hold for some $\lambda_i \in \mathbb{C}$ for which

$$\operatorname{rank} \begin{bmatrix} A - \lambda_i I & B \end{bmatrix} < n.$$

These λ_i are eigenvalues of A. Indeed, if PBH fails, $\exists \tilde{\eta}_i \neq 0$ such that

$$\tilde{\eta}_i' \begin{bmatrix} A - \lambda_i I & B \end{bmatrix} = 0 \iff \begin{cases} \tilde{\eta}_i' A = \lambda_i \tilde{\eta}_i' \\ \tilde{\eta}_i' B = 0 \end{cases}$$

i.e. this $\tilde{\eta}_i$ is a left eigenvector of A. λ_i's at which PBH fails called

- uncontrollable modes of (A, B).

Uncontrollable modes are eigenvalues of $A + BK$ for every K. Indeed,

$$\tilde{\eta}_i'(A + BK) = \tilde{\eta}_i' A = \lambda_i \tilde{\eta}_i',$$

which proves that λ_i is always an eigenvalue of $A + BK$.

Stabilizability

Pair (A, B) is said to be

- stabilizable if all its uncontrollable modes are stable (in open LHP).

Stabilizability means that there exists $K \in \mathbb{R}^{1 \times n}$ such that

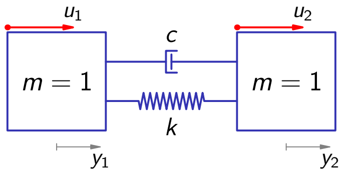
$$A_K := A + BK$$

is Hurwitz (all eigenvalues are in the open LHP).

Outline

Setup

Consider the following 2-mass system:



with external forces u_1 and u_2. It is described by the following equations:

$$\begin{bmatrix} \ddot{y}_1(t) \\ \ddot{y}_2(t) \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix} \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

State-space model

Possible realization (non-unique):

$$\begin{cases} \underbrace{\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \ddot{y}_1(t) \\ \ddot{y}_2(t) \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -c & c \\ k & -k & c & -c \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{C} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} \end{cases}$$

System has 4 modes (eigenvalues of A):

- $\lambda_1 = \lambda_2 = 0$ rigid body motion
- $\lambda_{3,4} = -c \pm \sqrt{c^2 - 2k}$ spring-damper dynamics

Controllability

If $u_1 = u$ and $u_2 = \beta u$ for some input u and constant β, then

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -c & c \\ k & -k & c & -c \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ \beta \end{bmatrix} u(t)$$

Controllability matrix:

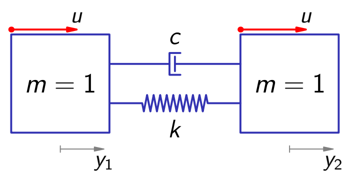
$$M_c = \begin{bmatrix} 0 & 1 & c(\beta-1) & -(2c^2-k)(\beta-1) \\ 0 & \beta & -c(\beta-1) & (2c^2-k)(\beta-1) \\ 1 & c(\beta-1) & -(2c^2-k)(\beta-1) & 4c(c^2-k)(\beta-1) \\ \beta & -c(\beta-1) & (2c^2-k)(\beta-1) & -4c(c^2-k)(\beta-1) \end{bmatrix}$$

with $\det M_c = -k^2(\beta^2-1)^2$. Thus, the system is

- uncontrollable for $\beta = \pm 1$.

What could it mean?

Controllability: $\beta = 1$ ($u_1 = u_2$)



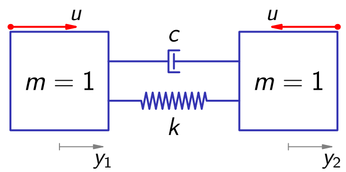
PBH test:

$$\operatorname{rank} \left. \begin{bmatrix} -\lambda & 0 & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 1 & 0 \\ -k & k & -c-\lambda & c & 1 \\ k & -k & c & -c-\lambda & 1 \end{bmatrix} \right|_{\lambda = -c \pm \sqrt{c^2-2k}} = 3,$$

(rank lost at uncontrollable modes of A). This agrees with our intuition that

- if equal forces applied to each mass, oscillations not excited.

Controllability: $\beta = -1$ $(u_1 = -u_2)$



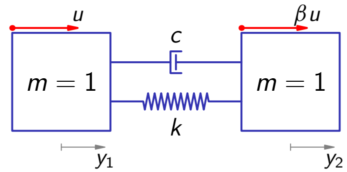
PBH test:

$$\operatorname{rank} \left.\begin{bmatrix} -\lambda & 0 & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 1 & 0 \\ -k & k & -c-\lambda & c & -1 \\ k & -k & c & -c-\lambda & 1 \end{bmatrix}\right|_{\lambda=0} = 3,$$

(rank lost at uncontrollable mode of A). This agrees with our intuition that

- if opposite forces applied to each mass, oscillations excited around the motion with zero acceleration.

Transfer functions for $u_1 = u$ and $u_2 = \beta u$



Transfer function from u to y_1:

$$P_1(s) = \frac{s^2 + c(\beta+1)s + k(\beta+1)}{s^2(s^2 + 2cs + 2k)}$$

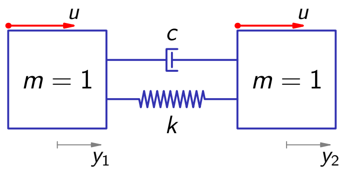
and transfer function from u to y_2:

$$P_2(s) = \frac{\beta s^2 + c(\beta+1)s + k(\beta+1)}{s^2(s^2 + 2cs + 2k)}$$

(both obtained via $C(sI - A)^{-1}B$).

Transfer functions for $u_1 = u$ and $u_2 = \beta u$ (contd)

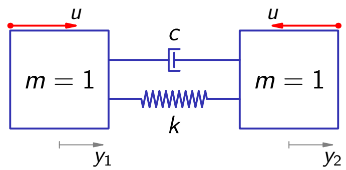
$\beta = 1$:



then

$$P_1(s) = P_2(s) = \frac{1}{s^2}.$$

$\beta = -1$:



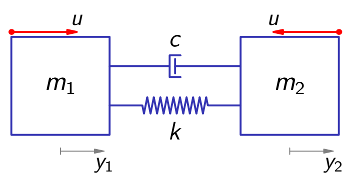
then

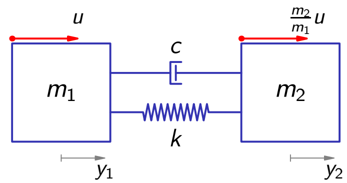
$$P_1(s) = -P_2(s) = \frac{1}{s^2 + 2cs + 2k}.$$

In both cases we have pole/zero cancellations (of different modes though).

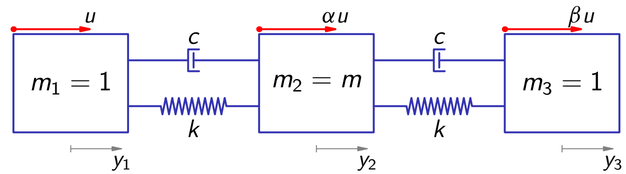
Extensions

- If masses are different, controllability is lost at $\beta = -1$ and $\beta = \frac{m_2}{m_1}$:





- If there are 3 masses, like in



the controllability is lost at

$$\beta = 1, \quad \alpha + \beta = -1, \quad \text{and} \quad \frac{2}{m}\alpha - \beta = 1$$

Try to explain...

Outline

Controllability

Example: 2-mass system (controllability)

Observability

Example: 2-mass system (observability)

Minimality and pole/zero cancellations

Controllability and observability of canonical realizations

Observability: definition

Consider

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) = Cx(t) + Du(t). \end{cases}$$

This system (or the pair (C, A)) is said to be

- observable if any initial state x_0 can be reconstructed from time history of $u(t)$ and $y(t)$ in interval $[0, t_1]$ for every $t_1 > 0$ and $u(t)$.

Simplifying observation:

- Without loss of generality we can assume that $u(t) \equiv 0$. Indeed, as

  $$y(t) = Ce^{At}x_0 + Du(t) + C\int_0^t e^{A(t-s)}Bu(s)\mathrm{d}s,$$

  x_0 reconstructable from time history of $y(t), u(t)$ iff it reconstructable from time history of $\tilde{y}(t) := y(t) - Du(t) - C\int_0^t e^{A(t-s)}Bu(s)\mathrm{d}s$.

Observability and observability matrix

Matrix

$$M_o := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

called the observability matrix.

Theorem

Pair (C, A) is observable if and only if $\det M_o \neq 0$.

Proof

If $u(t) \equiv 0$, then $y(t) = Ce^{At}x_0$ and

$$\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} = M_o x_0.$$

We have:

1. If $\det M_o \neq 0$, x_0 can be obtained from $n-1$ derivatives of $y(t)$ at 0.
2. If $\det M_o = 0$, then $\exists v \neq 0$ such that $M_o v = 0$, i.e. that $CA^i v = 0$ for all $i = 0, \dots, n-1$. Then, by Cayley-Hamilton,

   $$CA^i v = 0, \quad \forall i \in \mathbb{Z}^+ \implies Ce^{At}v \equiv 0.$$

   Therefore, if $x_0 = v$, then $y(t) = Ce^{At}x_0 \equiv 0$ and this initial condition is indistinguishable from $x(0) = 0$. $\square$

Observability and similarity

If $\hat{A} = TAT^{-1}$ and $\hat{C} = CT^{-1}$ for some nonsingular T, then

$$\hat{M}_o := \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \\ \vdots \\ \hat{C}\hat{A}^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} T^{-1}$$
$$= M_o T^{-1}$$

i.e.

- observability is not affected by similarity transformations.

Ovservability: some other tests

Theorem

The following statements are equivalent:

1. (C, A) *is observable;*
2. $\det M_o \neq 0$;
3. $\det W_o(t) \neq 0$ *for all* $t > 0$, *where* $W_o(t) := \int_0^t e^{A's} C' C e^{As} ds \in \mathbb{R}^{n \times n}$;
4. $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \in \mathbb{C}^{n+1 \times n}$ *has full column rank* $\forall \lambda \in \mathbb{C}$ *(PBH test);*
5. $C\eta \neq 0$ *for every right eigenvector* η *of* A;
6. *eigenvalues of* $A + LC$ *can be freely assigned by* $L \in \mathbb{R}^n$;
7. (A', C') *is controllable.*

The last statement shows

- duality between observability and controllability properties.

Ovservability: some useful facts

The following observations/definitions are important:

- $W_o(t)$-test leads to a derivative-free reconstruction algorithm. Let

$$\hat{x}(t) := [W_o(t_1)]^{-1} \int_0^t e^{A's} C' y(s) ds.$$

  In this case

$$\hat{x}(t_1) = [W_o(t_1)]^{-1} \int_0^{t_1} e^{A's} C' C e^{As} x_0 ds = x_0.$$

- If (C, A) is not observable, the PBH test fails for some $\lambda_i \in \mathbb{C}$. These λ_i are eigenvalues of A and called unobservable modes of (C, A).
- If λ is an unobservable mode of (C, A), then it is eigenvalue of $A + LC$ for any L.

Detectability

Pair (C, A) is said to be

- detectable if all its unobservable modes are stable (in open LHP).

Detectability means that there exists $L \in \mathbb{R}^n$ such that

$$A_L := A + LC$$

is Hurwitz (all eigenvalues are in the open LHP).

Outline

Controllability

Example: 2-mass system (controllability)

Observability

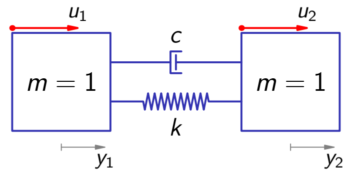
Example: 2-mass system (observability)

Minimality and pole/zero cancellations

Controllability and observability of canonical realizations

Setup

Consider again



with

$$\begin{cases} \underbrace{\begin{bmatrix} \dot y_1(t) \\ \dot y_2(t) \\ \ddot y_1(t) \\ \ddot y_2(t) \end{bmatrix}}_{\dot x(t)} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -c & c \\ k & -k & c & -c \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot y_1(t) \\ \dot y_2(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{C} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot y_1(t) \\ \dot y_2(t) \end{bmatrix} \end{cases}$$

Observability

If $y = y_1 + \gamma y_2$ for some γ, then output equation reads

$$y(t) = \begin{bmatrix} 1 & \gamma & 0 & 0 \end{bmatrix} x(t)$$

Observability matrix:

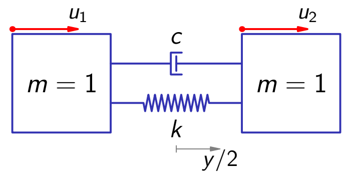
$$M_o = \begin{bmatrix} 1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & \gamma \\ k(\gamma-1) & -k(\gamma-1) & c(\gamma-1) & -c(\gamma-1) \\ -2ck(\gamma-1) & 2ck(\gamma-1) & (k-2c^2)(\gamma-1) & -(k-2c^2)(\gamma-1) \end{bmatrix}$$

with $\det M_o = -k^2(\gamma^2-1)^2$. Thus, the system is

- unobservable for $\gamma = \pm 1$.

What could it mean?

Observability: $\gamma = 1$ ($y = y_1 + y_2$)



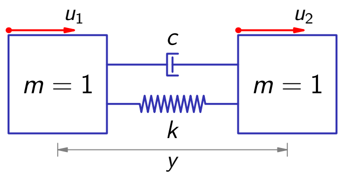
PBH test:

$$\operatorname{rank} \left. \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -k & k & -c-\lambda & c \\ k & -k & c & -c-\lambda \\ 1 & 1 & 0 & 0 \end{bmatrix} \right|_{\lambda = -c \pm \sqrt{c^2 - 2k}} = 3,$$

(rank lost at unobservable modes of A). This agrees with our intuition that

- oscillations cannot be seen via the center of mass.

Observability: $\gamma = -1$ ($y = y_1 - y_2$)



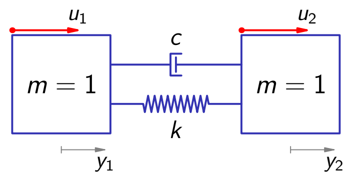
PBH test:

$$\operatorname{rank} \left. \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -k & k & -c-\lambda & c \\ k & -k & c & -c-\lambda \\ 1 & -1 & 0 & 0 \end{bmatrix} \right|_{\lambda=0} = 3,$$

(rank lost at unobservable mode of A). This agrees with our intuition that

– rigid body motion cannot be seen via relative position of the masses.

Transfer functions for $y = y_1 + \gamma y_2$



Transfer function from u_1 to y:

$$P_1(s) = \frac{s^2 + c(\gamma+1)s + k(\gamma+1)}{s^2(s^2 + 2cs + 2k)}$$

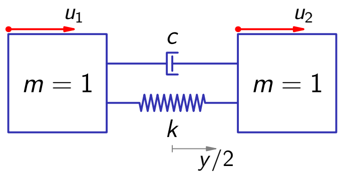
and transfer function from u_2 to y:

$$P_2(s) = \frac{\gamma s^2 + c(\gamma+1)s + k(\gamma+1)}{s^2(s^2 + 2cs + 2k)}$$

(both obtained via $C(sI - A)^{-1}B$).

Transfer functions for $y = y_1 + \gamma y_2$ (contd)

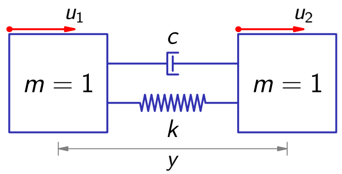
$\gamma = 1$:



then

$$P_1(s) = P_2(s) = \frac{1}{s^2}.$$

$\gamma = -1$:



then

$$P_1(s) = -P_2(s) = \frac{1}{s^2 + 2cs + 2k}.$$

In both cases we have pole/zero cancellations (of different modes though).

Outline

Minimal state-space realization

Example

Let $G(s) = \frac{1}{s+1}$. The following are its state-space realizations:

$$\begin{cases} \dot{x} = -x + u, \quad x(0) = 0 \\ y = x \end{cases} \quad \text{and} \quad \begin{cases} \dot{\hat{x}} = -\left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]\hat{x} + \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]u, \quad \hat{x}(0) = 0, \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}. \end{cases}$$

The first of them has state dimension $n = 1$, while the second one—$n = 2$. This indicates that there is *redundancy* in $\hat{x}$ (it accumulates somebody else history as well).

We may be interested to avoid redundancy. To this end, the notion of

- minimal state-space realization, i.e. a realization with minimal possible dimension,

plays a key role.

Minimality criterion

Theorem

Realization

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \\ y(t) = Cx(t) + Du(t) \end{cases}$$

is minimal iff it is both controllable and observable.

Explanations:

- uncontrollable part of x cannot be affected by input $u(t)$,
- unobservable part of x is invisible from output $y(t)$.

Important fact:

- every two minimal realizations of the same system are similar

(i.e. there is a similarity transformation between them).

Minimality and poles

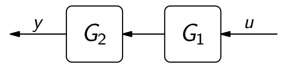
Theorem

If

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \\ y(t) = Cx(t) + Du(t) \end{cases}$$

is minimal, then $\lambda \in \mathbb{C}$ *is a pole of* $G(s) = D + C(sI - A)^{-1}B$ *iff it is an eigenvalue of* A.

Minimality of the cascade connection



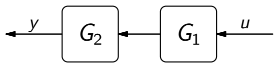
We already saw that the state-space realization of $G = G_2 G_1$ is

$$\begin{cases} \begin{bmatrix} \dot{x}_2(t) \\ \dot{x}_1(t) \end{bmatrix} = \begin{bmatrix} A_2 & B_2 C_1 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} C_2 & 0 \end{bmatrix} \begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix} \end{cases}$$

(the order of the state is swapped comparing to what we saw in Lecture 6). The question is:

- if the realizations of G_1 and G_2 are minimal, when so is that of G?

Pole-zero cancellations & state space: I



Let α be pole of $G_1(s)$ and not pole of $G_2(s)$. Assuming that all realizations are minimal, α is an eigenvalue of A_1 and not that of A_2. Define

$$v := \begin{bmatrix} (\alpha I - A_2)^{-1} B_2 C_1 \\ I \end{bmatrix} v_\alpha, \quad \text{where } (\alpha I - A_1) v_\alpha = 0$$

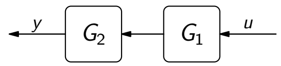
(note that minimality means that $C_1 v_\alpha \neq 0$). Then,

$$\begin{bmatrix} A_2 & B_2 C_1 \\ 0 & A_1 \end{bmatrix} v = \begin{bmatrix} (A_2(\alpha I - A_2)^{-1} + I) B_2 C_1 \\ A_1 \end{bmatrix} v_\alpha = \alpha v$$

and

$$\begin{bmatrix} C_2 & 0 \end{bmatrix} v = C_2 (\alpha I - A_2)^{-1} B_2 C_1 v_\alpha = G_2(\alpha) C_1 v_\alpha.$$

Pole-zero cancellations & state space: I (contd)



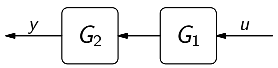
Thus,

$$\begin{bmatrix} C_2 & 0 \end{bmatrix} v = 0 \iff G_2(\alpha) = 0$$

and α is unobservable mode of $G_2 G_1$ iff it is zero of $G_2(s)$. In other words,

– any mode of G_1 is unobservable from y iff it's canceled by a zero of G_2.

Pole-zero cancellations & state space: II



Let β be pole of $G_2(s)$ and not pole of $G_1(s)$. Assuming that all realizations are minimal, β is an eigenvalue of A_2 and not that of A_1. Define

$$v' := v'_\beta \begin{bmatrix} I & B_2 C_1 (\beta I - A_1)^{-1} \end{bmatrix}, \quad \text{where } v'_\beta (\beta I - A_2) = 0$$

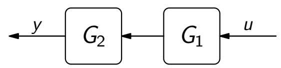
(note that minimality means that $v'_\beta B_2 \neq 0$). Then,

$$v' \begin{bmatrix} A_2 & B_2 C_1 \\ 0 & A_1 \end{bmatrix} = \begin{bmatrix} v'_\beta A_2 & v'_\beta B_2 C_1 (I + (\beta I - A_1)^{-1} A_1) \end{bmatrix} = \beta v'$$

and

$$v' \begin{bmatrix} 0 \\ B_1 \end{bmatrix} = v'_\beta B_2 C_1 (\beta I - A_1)^{-1} B_1 = v'_\beta B_2 G_1(\beta).$$

Pole-zero cancellations & state space: II (contd)



Thus,

$$v' \begin{bmatrix} 0 \\ B_1 \end{bmatrix} = 0 \iff G_1(\beta) = 0$$

and β is uncontrollable mode of $G_2 G_1$ iff it is zero of $G_1(s)$. In other words,

– any mode of G_2 is uncontrollable by u iff it's canceled by a zero of G_1.

Outline

Controllability

Example: 2-mass system (controllability)

Observability

Example: 2-mass system (observability)

Minimality and pole/zero cancellations

Controllability and observability of canonical realizations

Controllability of the companion form

In this case

$$A = A_{\rm cf} := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \quad \text{and} \quad B = B_{\rm cf} := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

and we have:

Theorem
A realization in the companion form is always controllable.

Remark: It can be shown that

$$M_{\rm c,cf} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}^{-1} \implies \det M_{\rm c,cf} = (-1)^{\lfloor n/2 \rfloor}.$$

Controllability of the companion form: proof

If $K = \begin{bmatrix} k_0 & k_1 & \cdots & k_{n-1} \end{bmatrix}$, then

$$\begin{aligned} A_K &:= A_{\rm cf} + B_{\rm cf} K \\ &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_0 & k_1 & \cdots & k_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -(a_0 - k_0) & -(a_1 - k_1) & \cdots & -(a_{n-1} - k_{n-1}) \end{bmatrix} \end{aligned}$$

is still a companion form. Hence,

$$\chi_{A_K}(s) = s^n + (a_{n-1} - k_{n-1}) s^{n-1} + \cdots + (a_1 - k_1) s + (a_0 - k_0)$$

can be made arbitrary by a choice of $K \implies$ controllability. □

Observability of the observer form

In this case

$$A = A_{\rm of} := \begin{bmatrix} -a_{n-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & \cdots & 1 \\ -a_0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad C = C_{\rm of} := \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

and we have (prove it yourselves):

Theorem
A realization in the observer form is always observable.

Remark: It can be shown that

$$M_{\rm o,of} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & \cdots & 1 & 0 \\ a_1 & a_2 & \cdots & a_{n-1} & 1 \end{bmatrix}^{-1} \implies \det M_{\rm o,of} = 1.$$