Control Theory (00350188)
lecture no. 2

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Outline

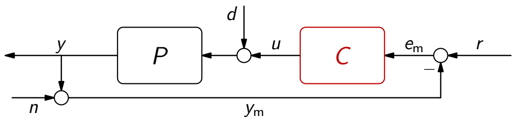
Reference signals in setpoint tracking problems

Reference profile: fastest realistic response and S-curves

Reference profile: fastest response under voltage constraints in DC motor

Anti-windup control

Step reference



By considering $r = y_f \mathbb{1}$ we express the steady-state goal to

- reach the final setpoint $\lim_{t\to\infty} y(t) = y_f$.

Step r is also used as a test signal to characterize quality of transients, e.g.

- overshoot
- raise time
- settling time

in controller design. This is convenient (analysis simplified / standardized). But

- does it make sense to use steps as *actual* reference signals ?
- can we do better via different r even if the final goal is a setpoint ?

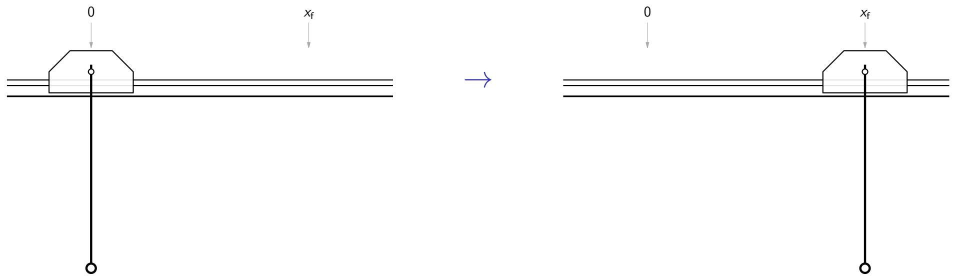
Example 1: moving cart with pendulum

Consider an undamped pendulum on a cart. The control input is the cart position x, the output is the pendulum angle θ. The linearized plant

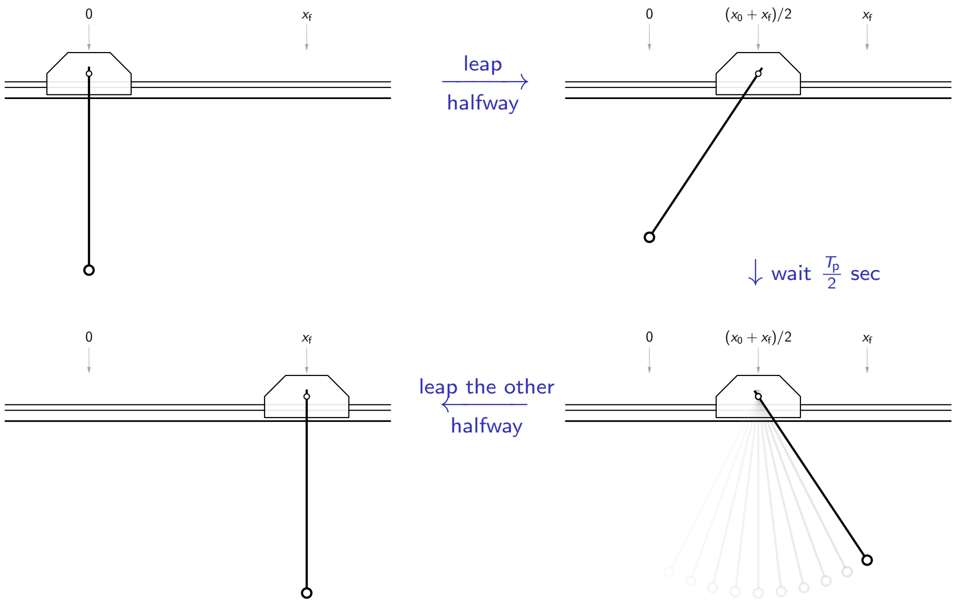
$$P(s) = \frac{s^2}{ls^2 + g},$$

where l is the pendulum length (so period $T_p := 2\pi\sqrt{l/g}$). Our goal is to

- move the cart *quickly* from $x = 0$ to x_f w/o oscillating the pendulum.



Example 1: posicast control (by Otto J. M. Smith)



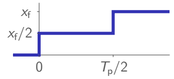
Example 1: moral

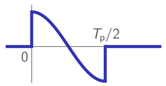
There is

- more than the (smoothened) step reference

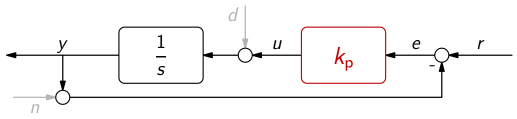
and

- transients can be improved by an elaborate choices of the command signal:

$x(t) =$ 

$\Longrightarrow$ $\theta(t) =$ 

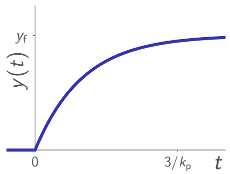
Example 2

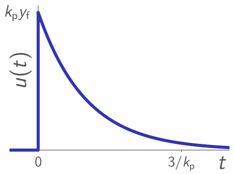


with

$$T(s) = \frac{k_p}{s + k_p} \quad \text{and} \quad T_c(s) = \frac{k_p s}{s + k_p}.$$

and the 5% settling time $t_s \approx 3\tau = 3/k_p$ independent of the setpoint y_f. If $r = y_f\mathbb{1}$, then, by linearity, both y and u are proportional to y_f:





meaning that

- large setpoint changes might cause actuator "overflow".

Example 2: moral

When faces real-world limitations,

- linearity sucks

in the choice of the reference signal. Even the choice

$$r = T_{\text{ref}} y_f \mathbb{1}$$

for a low-pass T_{ref} that smoothens the reference signal won't resolve that.

Outline

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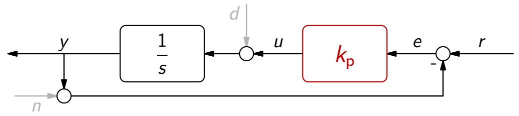
Reference profile: fastest response under voltage constraints in DC motor

Anti-windup control

What's wrong here ?



The problem is that the

- step reference is not realistic for inertial systems,

no inertial system can be expected to jump under a limited input. As such, the more successful we are in following such a command, the less affordable the price is.

For a problem to be realistic (resources are always limited),

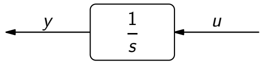
- t_s should depend on the setpoint change

and, in control terms, we need a

- nonlinear dependence of r on the setpoint.

Realistic settling time

Consider



under constraint $|u(t)| \leq u_{\max}$. Observe that

- t_s decreases as $|\dot{y}|$ increases
- $|\dot{y}(t)| = |u(t)|$
- y stops immediately as $u = 0$

Thus, $|\dot{y}(t)| \leq u_{\max}$ and the shortest t_s requires $|u(t)| = u_{\max}$ till $y(t) = y_f$. Hence,

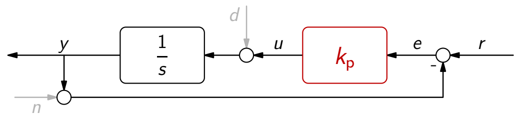
$$t_s \geq t_{s,\min} := |y_f|/u_{\max}.$$

This bound depends on the setpoint y_f (and on $u_{\max}$) and is attained via

$$u(t) = \begin{cases} \operatorname{sign}(y_f)u_{\max} & \text{if } t \leq t_{s,\min} \\ 0 & \text{if } t > t_{s,\min} \end{cases} \quad \text{and} \quad y(t) = \begin{cases} \operatorname{sign}(y_f)u_{\max}t & \text{if } t \leq t_{s,\min} \\ y_f & \text{if } t \geq t_{s,\min} \end{cases}$$

which are nonlinear functions of y_f.

Unity-feedback workaround

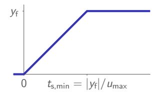


Let's pick r that yields

- the fastest system response under given physical constraints,

which in our case results in

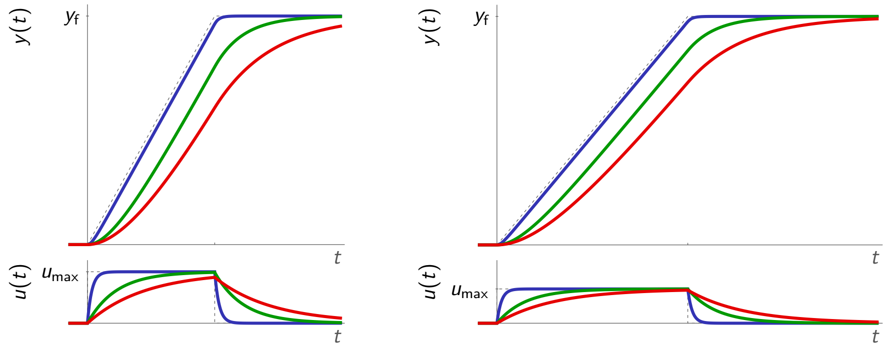
$$r(t) = \begin{cases} \operatorname{sign}(y_f)u_{\max}t & \text{if } t \leq t_{s,\min} \\ y_f & \text{if } t \geq t_{s,\min} \end{cases} =$$



instead of $r = y_f\mathbb{1}$.

Unity-feedback: simulation results

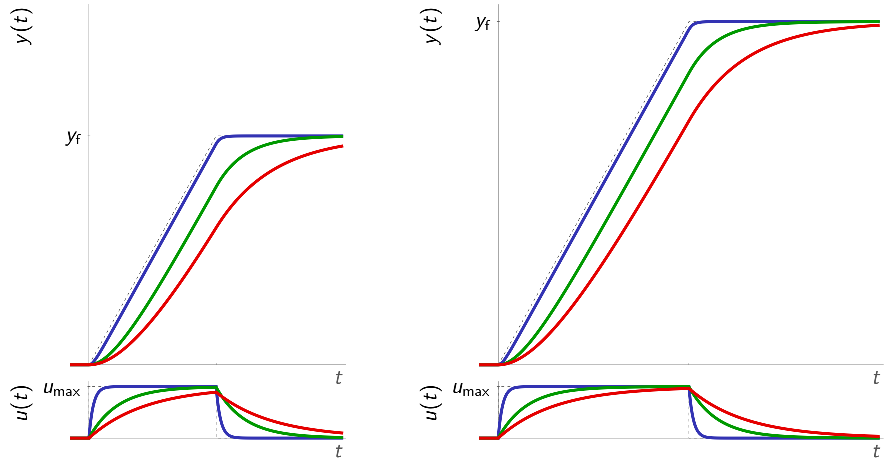
For controller gains $k_p = 13$, $k_p = 2$, and $k_p = 1$ and two different u_{max}'s:



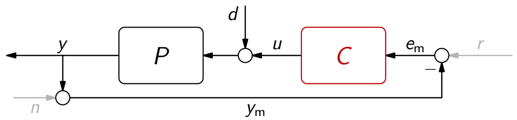
- r now agrees with the physics / limitations of the system
  hence, the resulting control signal is within the limits for all k_p
- tracking properties still depend on k_p (i.e. on closed-loop bandwidth)
  it's our job to pick agreeing k_p and r

Unity-feedback: simulation results (contd)

For controller gains $k_p = 13$, $k_p = 2$, and $k_p = 1$ and two different y_f's:



General considerations



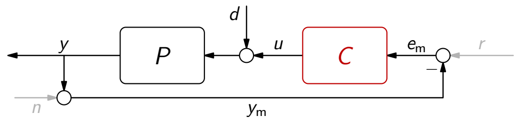
Ideally, pick r that yields

- the fastest system response under given physical constraints.

But this might be rather knotty for

- more complex dynamics
  even for 2-order systems solution more complicated; no analytic solution in general
- more complex constraints
  might involve internal signals, like DC motor current, sensor limitations, et cetera
- nonzero initial conditions
  e.g. if a new setpoint arrives before the previous one was reached

Pragmatic alternative



Pick r that yields

- the fastest trajectory under given constraints on derivatives of r.

For example,

$$\begin{aligned} &\text{minimize} && t_f \\ &\text{subject to} && r(0) = 0, \quad r(t_f) = y_f, \quad \dot r(t_f) = 0, \quad \ddot r(t_f) = 0, \ldots \\ & && |\dot r(t)| \le v_{max} \\ & && |\ddot r(t)| \le a_{max} \\ & && |\dddot r(t)| \le j_{max} \end{aligned}$$

for given $v_{max} > 0$ (velocity), $a_{max} > 0$ (acceleration), and $j_{max} > 0$ (jerk), which indirectly reflect physical constraints, and a given setpoint y_f.

Example: constraints on velocity and acceleration

Problem:

$$
\begin{aligned}
&\text{minimize} && t_\mathrm{f}\\
&\text{subject to} && r(0)=0, \quad r(t_\mathrm{f})=y_\mathrm{f}, \quad \dot r(t_\mathrm{f})=0\\
& && |\dot r(t)|\le v_\mathrm{max}, \quad |\ddot r(t)|\le a_\mathrm{max}
\end{aligned}
$$

for given $v_\mathrm{max}>0$, $a_\mathrm{max}>0$, and y_f.

Complications (due to $a_\mathrm{max}<\infty$):

- maximal velocity cannot be achieved from the beginning
- r cannot be stopped immediately if its velocity is nonzero

Strategy:

1. start with maximal acceleration / stop with maximal deceleration
2. this might be sufficient (if y_f is so small that v_max is not reached)
3. if not, reset acceleration at $t=t_\mathrm{sw1}$, where $|\dot r(t_\mathrm{sw1})|=v_\mathrm{max}$ is satisfied, then start deceleration at $t=t_\mathrm{sw2}$, for which $r(t_\mathrm{sw2})=y_\mathrm{f}-r(t_\mathrm{sw1})$

Example: some calculations

1. Maximal acceleration (assume, for simplicity, that $y_\mathrm{f}>0$):

$$\ddot r(t)=a_\mathrm{max} \implies \dot r(t)=a_\mathrm{max}t \implies r(t)=a_\mathrm{max}t^2/2.$$

Then $r(t)=y_\mathrm{f}/2$ at $t_\mathrm{sw}=\sqrt{y_\mathrm{f}/a_\mathrm{max}}$, so that

$$t_\mathrm{f}=2\sqrt{y_\mathrm{f}/a_\mathrm{max}} \quad \text{and} \quad \dot r(t_\mathrm{sw})=\sqrt{y_\mathrm{f}\,a_\mathrm{max}}.$$

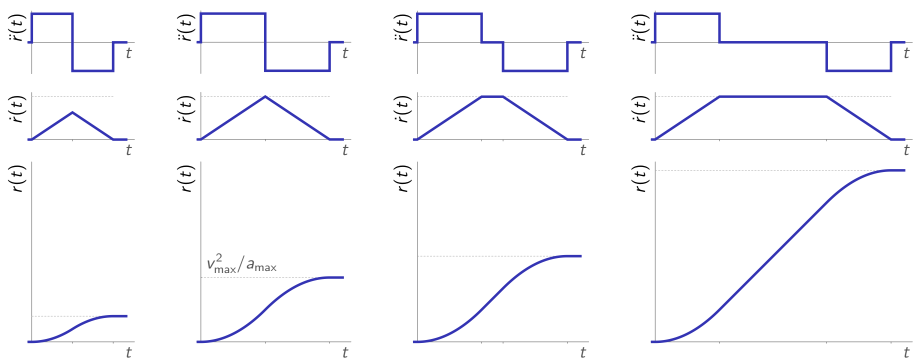
2. This strategy suffices iff $\sqrt{y_\mathrm{f}\,a_\mathrm{max}}\le v_\mathrm{max} \iff y_\mathrm{f}\le v_\mathrm{max}^2/a_\mathrm{max}$.

3. The first switch is at $\dot r(t_\mathrm{sw1})=v_\mathrm{max}$, therefore $t_\mathrm{sw1}=v_\mathrm{max}/a_\mathrm{max}$. At this moment $r(t_\mathrm{sw1})=v_\mathrm{max}^2/(2a_\mathrm{max})<y_\mathrm{f}/2$ and continues linearly, as

$$r(t)=v_\mathrm{max}^2/(2a_\mathrm{max})+v_\mathrm{max}(t-t_\mathrm{sw1})=v_\mathrm{max}t-v_\mathrm{max}^2/(2a_\mathrm{max}).$$

The second switch happens at $r(t_\mathrm{sw2})=y_\mathrm{f}-v_\mathrm{max}^2/(2a_\mathrm{max})$, from which $t_\mathrm{sw2}=y_\mathrm{f}/v_\mathrm{max}$. Finally, because of symmetry $t_\mathrm{f}=t_\mathrm{sw1}+t_\mathrm{sw2}$.

Reference trajectories (S-curve profiles)



with the settling time:

$$t_\mathrm{s}=\begin{cases}2\sqrt{|y_\mathrm{f}|/a_\mathrm{max}} & \text{if } y_\mathrm{f}\le v_\mathrm{max}^2/a_\mathrm{max}\\ |y_\mathrm{f}|/v_\mathrm{max}+v_\mathrm{max}/a_\mathrm{max} & \text{if } y_\mathrm{f}\ge v_\mathrm{max}^2/a_\mathrm{max}\end{cases}$$

and switches at $t_\mathrm{sw}=\sqrt{|y_\mathrm{f}|/a_\mathrm{max}}$ or $t_\mathrm{sw1}=v_\mathrm{max}/a_\mathrm{max}$ and $t_\mathrm{sw2}=|y_\mathrm{f}|/v_\mathrm{max}$.

Outline

Preliminaries: residues, simple poles case

Let $G(s)$ have a *simple* pole at $s = a$. The residue of G at $s = a$,

$$\operatorname{Res}(G(s), a) := \lim_{s \to a} (s - a) G(s).$$

If $\operatorname{Res}(G(s), a) = 0$, then the singularity at $s = a$ is removable.

If $G(s)$ is rational, proper, and has only simple poles, at $s = s_i$, then

$$G(s) = G(\infty) + \sum_{i=1}^{n} \frac{\operatorname{Res}(G(s), s_i)}{s - s_i}$$

(partial fraction expansion).

Preliminaries: residues, simple poles case (contd)

Example

$$G(s) = \frac{s^2}{ls^2 + g} \quad \Longrightarrow \quad G(s) = \frac{1}{l} + \frac{\mathrm{j}\sqrt{g/(4l^3)}}{s - \mathrm{j}\sqrt{g/l}} - \frac{\mathrm{j}\sqrt{g/(4l^3)}}{s + \mathrm{j}\sqrt{g/l}}$$

Example

The function

$$G(s) = \frac{1 - \alpha \mathrm{e}^{-\tau s}}{s}$$

has a single singularity at $s = 0$.

$$\operatorname{Res}(G(s), 0) = \lim_{s \to 0} sG(s) = \lim_{s \to 0} (1 - \alpha \mathrm{e}^{-\tau s}) = 1 - \alpha.$$

Two cases:

$\alpha \neq 1$ $\operatorname{Res}(G(s), 0) \neq 0$ and the singularity is a pole

$\alpha = 1$ $\operatorname{Res}(G(s), 0) = 0$ and the singularity is removable (not a pole)

Preliminaries: a special complex function

Let

$$G(s) = \frac{N_0(s) + N_1(s)\mathrm{e}^{-\tau_1 s} + \cdots + N_n(s)\mathrm{e}^{-\tau_n s}}{D(s)} \qquad (*)$$

for polynomials $D(s)$ and $N_i(s)$ such that

- $\deg D(s) \geq \deg N_i(s)$ for all $i = 0, \ldots, n$,
- all roots s_i of $D(s)$ are simple,

and $0 < \tau_1 < \tau_2 < \cdots < \tau_n$. Expand, for $j = 0, \ldots, n$,

$$\frac{N_j(s)}{D(s)} = \beta_j + \sum_i \frac{\alpha_{ij}}{s - s_i}, \quad \alpha_{ij} = \operatorname{Res}\Bigl(\frac{N_j(s)}{D(s)}, s_i\Bigr) \text{ and } \beta_j = \lim_{s \to \infty} \frac{N_j(s)}{D(s)}.$$

Hence

$$G(s) = \beta(s) + \sum_i \frac{\alpha_i(s)}{s - s_i}, \quad \alpha_i(s) := \sum_{j=0}^{n} \alpha_{ij}\mathrm{e}^{-\tau_j s} \text{ and } \beta(s) := \sum_{j=0}^{n} \beta_j \mathrm{e}^{-\tau_j s}.$$

Note that $\alpha_i(s_i) = \operatorname{Res}(G(s), s_i)$.

Preliminaries: impulse response of $(*)$

Let

$$G_i(s) := \frac{\alpha_i(s)}{s - s_i} = \sum_{j=0}^{n} \frac{\alpha_{ij}}{s - s_i} \mathrm{e}^{-\tau_j s}.$$

Its inverse Laplace transform

$$g_i(t) = \sum_{j=0}^{n} \alpha_{ij} \mathrm{e}^{s_i(t - \tau_j)} \mathbb{1}(t - \tau_j) = \mathrm{e}^{s_i t} \sum_{j=0}^{n} \alpha_{ij} \mathrm{e}^{-s_i \tau_j} \mathbb{1}(t - \tau_j)$$

If $t > \tau_n$, then $\mathbb{1}(t - \tau_j) = 1$ for all j and

$$g_i(t) = \mathrm{e}^{s_i t} \sum_{j=0}^{n} \alpha_{ij} \mathrm{e}^{-s_i \tau_j} = \mathrm{e}^{s_i t} \alpha_i(s_i) \overset{\alpha_i(s_i)=0}{=} 0, \qquad \forall t > \tau_n.$$

Hence,

- $\alpha_i(s_i) = 0, \ \forall i \implies \operatorname{supp}(g) \subset [0, \tau_n]$ and G is BIBO stable.

Systems, whose impulse responses have support over finite intervals dubbed

- FIR (finite impulse response) systems.

Preliminaries: step response of $(*)$

The step response of G is

$$Y(s)=\frac{G(s)}{s}\iff y(t)=\int_0^t g(\theta)\mathrm{d}\theta.$$

If $\alpha_i(s_i)=0$ for all i, then $\operatorname{supp}(g)\subset[0,\tau_n]$ and

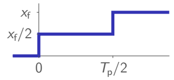
$$y(t)=\int_0^{\tau_n} g(\theta)\mathrm{d}\theta=\text{const}=G(0),\quad \forall t>\tau_n.$$

In other words, the

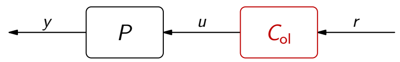
– step response of FIR systems converges to steady state in finite time.

Remark: posicast control revisited

Control

$x(t)=$  $\implies X(s)=\dfrac{1+\mathrm{e}^{-sT_\mathrm{p}/2}}{2}\dfrac{x_\mathrm{f}}{s}$

corresponds to the open-loop architecture (with $u=x$ and $y=\theta$)



under

– $P(s)=\dfrac{s^2}{ls^2+g}$ and stable $C_\mathrm{ol}(s)=\dfrac{1+\mathrm{e}^{-sT_\mathrm{p}/2}}{2}$,
– step reference $r=x_\mathrm{f}\mathbb{1}$

and the controlled system

$$P(s)C_\mathrm{ol}(s)=\frac{0.5s^2+0.5s^2\mathrm{e}^{-sT_\mathrm{p}/2}}{ls^2+g}$$

is of form $(*)$ and has $\alpha_i(s_i)=0$ for $i=1,2$, just because $C_\mathrm{ol}(\pm\mathrm{j}\sqrt{gl})=0$ (check it yourselves), so is FIR.

Remark: posicast control for dampened pendulum

Let

$$P(s)=\frac{s^2}{ls^2+2cs+g},\quad \text{for } 0\le c<\sqrt{gl}$$

with poles at $-\sigma\pm\mathrm{j}\omega$ for $\sigma=c/l$ and $\omega=\sqrt{gl-c^2}/l=2\pi/T_\mathrm{p}$. Choose

$$C_\mathrm{ol}(s)=\phi_0+\phi_1\mathrm{e}^{-\tau s}.$$

We shall require

– $C_\mathrm{ol}(0)=1=\phi_0+\phi_1$ — $x=x_\mathrm{f}$ is steady state
– $C_\mathrm{ol}(-\sigma\pm\mathrm{j}\omega)=0$ — posicast, i.e. FIR

Equivalent to

$$\begin{bmatrix}1 & 1\\ 1 & \mathrm{e}^{\tau\sigma}\cos(\tau\omega)\\ 0 & \mathrm{e}^{\tau\sigma}\sin(\tau\omega)\end{bmatrix}\begin{bmatrix}\phi_0\\ \phi_1\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

(because $\mathrm{e}^{-\tau(-\sigma\pm\mathrm{j}\omega)}=\mathrm{e}^{\tau\sigma}(\sin(\tau\omega)\mp\mathrm{j}\cos(\tau\omega))$ in $\phi_0,\phi_1\in\mathbb{R}$ and $\tau>0$.

Remark: posicast control for dampened pendulum (contd)

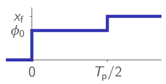
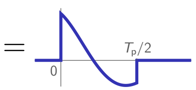
As $\phi_1\ne 0$ (otherwise unsolvable), must have $\sin(\tau\omega)=0$, with the shortest

$$\tau=\frac{\pi}{\omega}\implies\begin{bmatrix}1 & 1\\ 1 & -\mathrm{e}^{\tau\sigma}\end{bmatrix}\begin{bmatrix}\phi_0\\ \phi_1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\implies\begin{bmatrix}\phi_0\\ \phi_1\end{bmatrix}=\begin{bmatrix}\mathrm{e}^{\tau\sigma}\\ 1\end{bmatrix}\frac{1}{1+\mathrm{e}^{\tau\sigma}}$$

($\phi_0>1/2$ if $c>0$). Taking into account that $T_\mathrm{p}=2\pi/\omega$, we end up with

$$C_\mathrm{ol}(s)=\frac{\mathrm{e}^{0.5T_\mathrm{p}c/l}+\mathrm{e}^{-0.5T_\mathrm{p}s}}{1+\mathrm{e}^{0.5T_\mathrm{p}c/l}}.$$

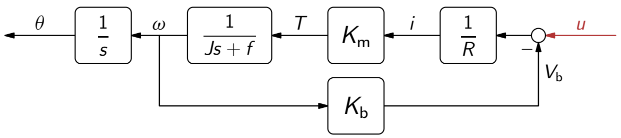
The resulting

$x(t)=$  $\implies \theta(t)=$ 

also finish the move in $T_\mathrm{p}/2$, but

– not posicast, in the sense that $\dot\theta(t)|_{t\uparrow T_\mathrm{p}/2}\ne 0$, whenever $c\ne 0$.

Fastest shaft angle change under voltage constraints



Consider the task of turning the shaft of a DC motor resting at $\theta(0) = \theta_0$ to a new angular position, say $\theta_f \neq \theta_0$, and resting there. We may need to

- do that as quick as possible under physical constraints.

A possible constraint[1] is the

- input voltage amplitude, $|u(t)| \leq u_{\max}$ for some $u_{\max} > 0$.

Our goal is to generate u that may then be a good choice for the reference trajectory r.

[1] The armature current amplitude is another, perhaps even more practical, possibility.

Mathematical formulation

Let θ satisfy

$$RJ\ddot{\theta}(t) + (Rf + K_m K_b)\dot{\theta}(t) = K_m u(t) \quad \iff \quad \tau\ddot{\theta}(t) + \dot{\theta}(t) = ku(t)$$

for $\tau := RJ/(Rf + K_m K_b)$ and $k := K_m/(Rf + K_m K_b)$,

$$\begin{aligned} &\text{minimize} && t_f \\ &\text{subject to} && \theta(0) = \theta_0, \quad \dot{\theta}(0) = 0, \quad \theta(t_f) = \theta_f, \quad \dot{\theta}(t_f) = 0 \\ &&& |u(t)| \leq u_{\max} \end{aligned}$$

for given θ_0, θ_f, and $u_{\max} > 0$. This problem depends on system dynamics. Note that the model in the Laplace variable domain,

$$\Theta(s) = \frac{\theta_0}{s} + \frac{k}{s(\tau s + 1)}U(s),$$

is affected by the initial condition.

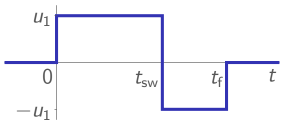
Time-optimal control

The studied problem is a special case of the time-optimal control problems, whose theory goes beyond the scope of this course. Outcomes of the theory relevant for the discussion below are:

- optimal $u(t)$ in $0 < t < t_f$ takes values only in the set $\{-u_{\max}, u_{\max}\}$ (such control strategy is known as bang-bang control)
- there is a finite number of switches $u_{\max} \rightleftharpoons -u_{\max}$ for any finite t_f
- if the plant has only real poles, say n, then the number of switches in $(0, t_f)$ is at most $n - 1$

Applying to our problem[2],

$$u(t) = \begin{cases} u_1 & \text{if } t \in (0, t_{sw}) \\ -u_1 & \text{if } t \in (t_{sw}, t_f) \\ 0 & \text{if } t \in (t_f, \infty) \end{cases} =$$



for $|u_1| = u_{\max}$ and some $0 < t_{sw} < t_f$ to be determined.

[2] Mind that $u(t) = 0$ whenever $t \notin [0, t_f]$ because of an integrator in the plant.

Solution logic

Thus, $u(t) = u_1\bigl(\mathbb{1}(t) - 2\mathbb{1}(t - t_{sw}) + \mathbb{1}(t - t_f)\bigr)$, or

$$U(s) = u_1\frac{1 - 2e^{-st_{sw}} + e^{-st_f}}{s},$$

and

$$\Theta(s) = \frac{\theta_0}{s} + \frac{ku_1(1 - 2e^{-st_{sw}} + e^{-st_f})}{s^2(\tau s + 1)} = \Biggl(\overbrace{\theta_0 + \frac{ku_1(1 - 2e^{-st_{sw}} + e^{-st_f})}{s(\tau s + 1)}}^{G_\theta(s)}\Biggr)\frac{1}{s}$$

Our goal is to

- determine $\text{sign}(u_1)$, t_{sw}, and $t_f > t_{sw}$

such that $G_\theta(s)$ is FIR and $G_\theta(0) = \theta_f$. This is equivalent[3] to

1. $\lim_{s\to 0} G_\theta(s) = \theta_f$
2. $\text{Res}(G_\theta(s), -1/\tau) = 0$

[3] Mind that the singularity of $G_\theta(s)$ at $s = 0$ is always removable, by construction.

Solution details

1. Condition $\lim_{s\to 0} G_\theta(s) = \theta_f$ reads

$$\theta_f = \theta_0 + \lim_{s\to 0} \frac{ku_1(1 - 2e^{-st_{sw}} + e^{-st_f})}{s(\tau s + 1)} = \theta_0 + ku_1(2t_{sw} - t_f).$$

   Hence,

$$ku_1(2t_{sw} - t_f) = \theta_f - \theta_0.$$

2. Condition $\mathrm{Res}(G_\theta(s), -1/\tau) = 0$ reads

$$0 = \lim_{s\to -1/\tau}\Bigl(s + \frac{1}{\tau}\Bigr) G_\theta(s) = \lim_{s\to -1/\tau} \frac{ku_1(1 - 2e^{-st_{sw}} + e^{-st_f})}{\tau s}$$
$$= -ku_1(1 - 2e^{t_{sw}/\tau} + e^{t_f/\tau}).$$

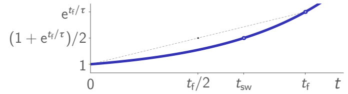
   Hence,

$$e^{t_{sw}/\tau} = \frac{1 + e^{t_f/\tau}}{2}.$$

Solution details (contd)

The equality

$$e^{t_{sw}/\tau} = \frac{1 + e^{t_f/\tau}}{2} \quad :$$



implies that $t_{sw} > t_f/2$, because

$$\frac{d}{dt} e^{t/\tau} = \frac{e^{t/\tau}}{\tau} > 0 \quad \text{and} \quad \frac{d^2}{dt^2} e^{t/\tau} = \frac{e^{t/\tau}}{\tau^2} > 0$$

for all t (meaning that $e^{t/\tau}$ is increasing and strictly convex). But then

$$(2t_{sw} > t_f) \wedge (ku_1(2t_{sw} - t_f) = \theta_f - \theta_0) \implies \mathrm{sign}(u_1) = \mathrm{sign}(\theta_f - \theta_0)$$

and

$$\frac{\theta_f - \theta_0}{u_1} = \frac{|\theta_f - \theta_0|}{u_{max}}.$$

Solution details (contd)

Thus, we end up with the following two equations for $t_{sw} > 0$ and $t_f > t_{sw}$:

$$2t_{sw} - t_f = \frac{|\theta_f - \theta_0|}{ku_{max}} \quad \text{and} \quad 2e^{t_{sw}/\tau} = 1 + e^{t_f/\tau}.$$

Hence, $t_f = 2t_{sw} - |\theta_f - \theta_0|/(ku_{max})$ and

$$e^{-|\theta_f - \theta_0|/(\tau k u_{max})}(e^{t_{sw}/\tau})^2 - 2e^{t_{sw}/\tau} + 1 = 0.$$

Solving this quadratic equation in $e^{t_{sw}/\tau}$ yields (take "+" to have $t_{sw} < t_f$)

$$t_{sw} = \frac{|\theta_f - \theta_0|}{ku_{max}} + \tau \ln\Bigl(1 + \sqrt{1 - e^{-|\theta_f - \theta_0|/(\tau k u_{max})}}\Bigr)$$

and

$$t_f = \frac{|\theta_f - \theta_0|}{ku_{max}} + 2\tau \ln\Bigl(1 + \sqrt{1 - e^{-|\theta_f - \theta_0|/(\tau k u_{max})}}\Bigr).$$

Both are increasing functions of $|\theta_f - \theta_0|$ and τ and decreasing of ku_{max}.

The fastest $\theta(t)$

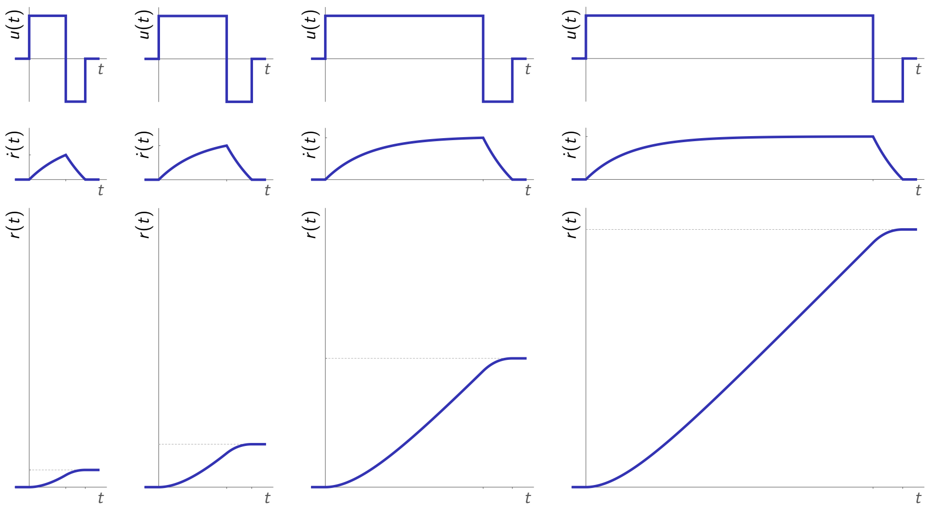
Taking the inverse Laplace transform of $\Theta(s)$, we finally get

$$\theta(t) = \begin{cases} \theta_0 + \bigl(t - (1 - e^{-t/\tau})\tau\bigr) ku_1 & \text{if } t \in [0, t_{sw}] \\ \theta_0 + \bigl(2t_{sw} + \tau - t - e^{-t/\tau}(2e^{t_{sw}/\tau} - 1)\tau\bigr) ku_1 & \text{if } t \in [t_{sw}, t_f] \\ \theta_f & \text{if } t \in [t_f, \infty) \end{cases}$$

where $u_1 = \mathrm{sign}(\theta_f - \theta_0)u_{max}$. The corresponding angular velocity

$$\omega(t) = \begin{cases} (1 - e^{-t/\tau})ku_1 & \text{if } t \in [0, t_{sw}] \\ \bigl(e^{-t/\tau}(2e^{t_{sw}/\tau} - 1) - 1\bigr) ku_1 & \text{if } t \in [t_{sw}, t_f] \\ 0 & \text{if } t \in [t_f, \infty) \end{cases}$$

Resulting reference trajectories



are reminiscent of S-curves, but are not symmetric (stopping is cheaper).

Outline

Reference signals in setpoint tracking problems

Reference profile: fastest realistic response and S-curves

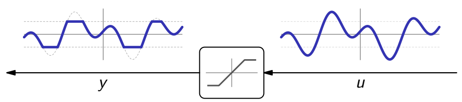
Reference profile: fastest response under voltage constraints in DC motor

Anti-windup control

Saturation

It is a system $u \mapsto y$, which we denote $\mathrm{sat}_{[a,b]}$, such that

$$y(t) = \begin{cases} a & \text{if } u(t) < a \\ u(t) & \text{if } a \le u(t) \le b \\ b & \text{if } u(t) > b \end{cases}$$

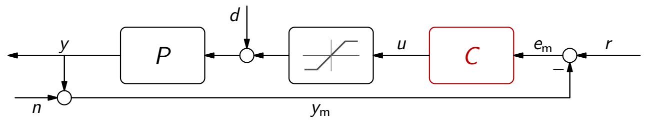


for given $a < b$. We use the short notation $\mathrm{sat}_a := \mathrm{sat}_{[-a,a]}$ for some $a > 0$. Think of a gas pedal in cars, water tap, integer overflow in computers, etc.

Saturation element is a nonlinear system (no superposition). Indeed,

$$\mathrm{sat}_1(2 \times 0.6 \sin t) \neq 2 \times \mathrm{sat}_1(0.6 \sin t) = 1.2 \sin t.$$

Saturation in feedback loop



All actuators saturate. Indeed,

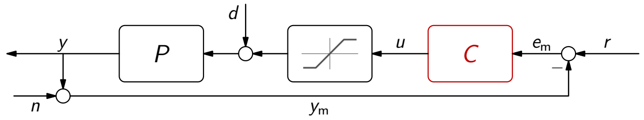
- force, torque,
- voltage, current,
- flow rate,
- ...

are ultimately limited. Some sensors saturate as well. We therefore must

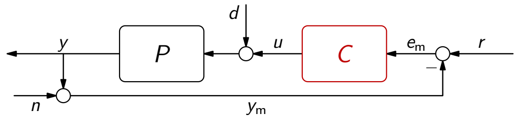
- respect the presence of (nonlinear) saturation elements

in any feedback loop.

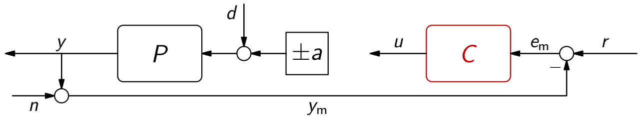
Saturation in feedback loop (contd)



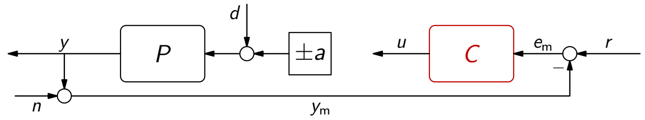
- if u does not saturate, it behaves as standard linear closed-loop system:



- if u saturates, it behaves as open-loop system:



Saturation in feedback loop (contd)



This doesn't help in general, yet it is

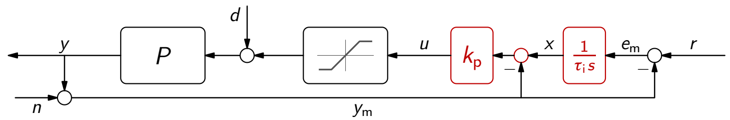
- especially problematic when either $P(s)$ or $C(s)$ is unstable.

What can be done:

- When plant is unstable, there is nothing we can do.
- Controllers are in our power, so
  1. if possible, it is advisable to avoid the use of unstable controllers;
  2. if not[4], controller should be modified when control signal saturates.

[4] E.g. the plant is not strongly stabilizable, an integral action is required, et cetera.

PI controllers and saturation



PI controller transforms y_{m} and r to control signal u according to

$$U(s) = \frac{k_{\mathrm{p}}}{\tau_{\mathrm{i}} s}\bigl(R(s) - Y_{\mathrm{m}}(s)\bigr) - k_{\mathrm{p}} Y_{\mathrm{m}}(s) \quad \text{or}^5 \quad \begin{cases} \dot{x}(t) = \frac{1}{\tau_{\mathrm{i}}}\bigl(r(t) - y_{\mathrm{m}}(t)\bigr) \\ u(t) = k_{\mathrm{p}}\bigl(x(t) - y_{\mathrm{m}}(t)\bigr) \end{cases}$$

While $u(t)$ saturates,

- state $x(t)$, acting in open loop, might accumulate a big value,

so that

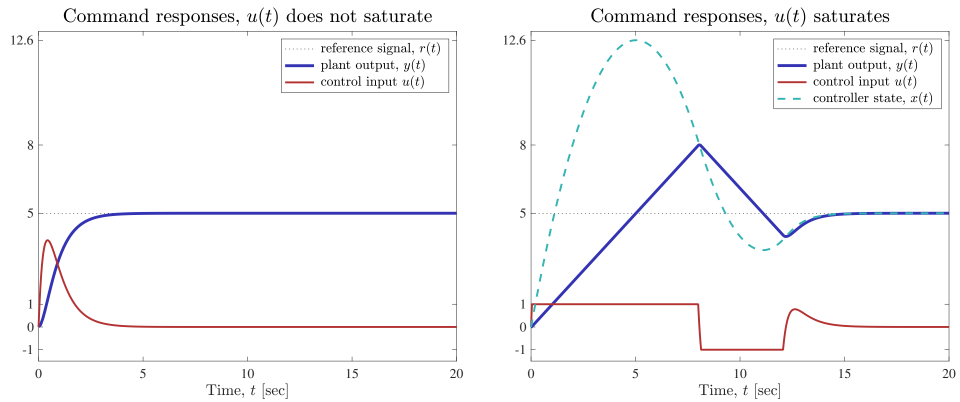
- u remains saturated even when $r - y_{\mathrm{m}}$ becomes small

(after all, dynamic systems have memory).

[5] This is how this controller is implemented.

PI controllers and saturation: example

Consider control system with $P(s) = 1/s$ and $C(s) = 5(1 + 1/s)$. We have:



State variable becomes very large by the time error approaches 0, hence

- y continues to grow until $x(t)$ becomes smaller than $y(t)$

(remember, $u = k_{\mathrm{p}}(x - y)$ and the direction of y equals the sign of $u = \dot{y}$).

Integrator windup

The effect of

- significant grow of the integrator state during actuator saturation

is called the integrator windup.

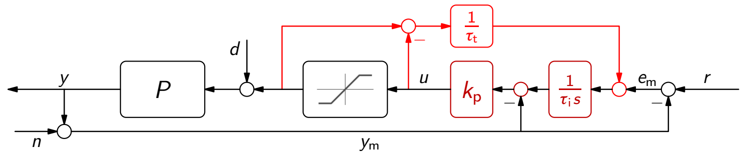
Arguably, most remedies for windup effect are based on

- preventing integrator state from unstable updating once u saturates.

Possible heuristics (sometimes equivalent):

- stop updating integrator when u saturates (conditional integration);
- implement integral action as interconnection of stable elements, with some of interconnections opened when u saturates;
- add internal controller feedback acting on $u - \mathrm{sat}_a(u)$.

Anti-windup scheme with internal feedback



This scheme (τ_t is called the tracking time constant) works as follows:

- if u does not saturate, then $\mathrm{sat}_a(u) - u = 0$ and it is a standard PI;
- if u saturates, then the controller becomes stable:

$$U(s) = k_p\left(\frac{1}{\tau_i s}\left(R(s) - Y_m(s) \pm \frac{a}{\tau_t} - \frac{1}{\tau_t}U(s)\right) - Y_m(s)\right),$$

so

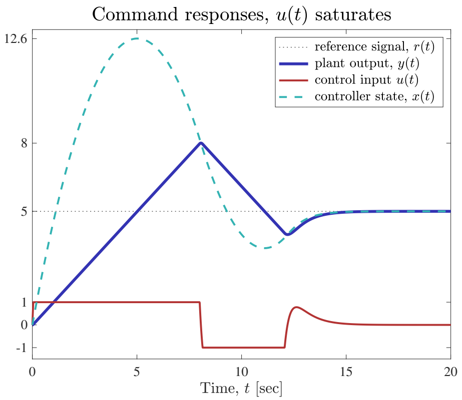
$$\frac{\tau_t\tau_i s + k_p}{\tau_t\tau_i s}\,U(s) = \frac{k_p}{\tau_i s}\left(R(s) - (\tau_i s + 1)Y_m(s) \pm \frac{a}{\tau_t}\right)$$

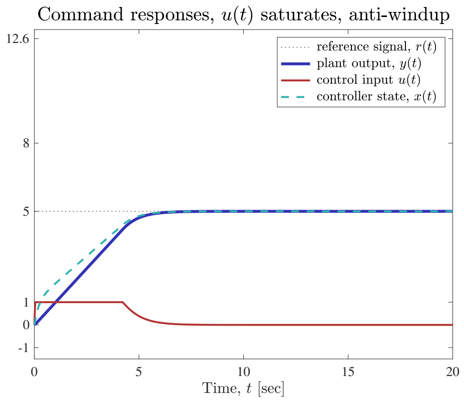
and then

$$U(s) = \frac{\tau_t k_p}{\tau_t\tau_i s + k_p}\,R(s) - \frac{\tau_t k_p(\tau_i s + 1)}{\tau_t\tau_i s + k_p}\,Y_m(s) \pm \frac{k_p}{\tau_t\tau_i s + k_p}\,a.$$

PI controllers and saturation: example (contd)

Internal feedback really helps (here $\tau_t = 1$):





Another anti-windup solution: saturation-aware r

In many situations we may

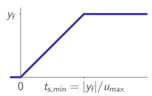
- avoid windup by a saturation-aware choice of the reference signal,

so no need in smart solutions to problems one shouldn't have gotten into in the first place.

Example:

With $P(s) = \frac{1}{s}$ and $|u(t)| \le 1$ we have no chance to raise faster than $y(t) = t$ anyway. It may make sense to pick

$$r(t) = \begin{cases} t & \text{if } t \le r_f \\ r_f & \text{if } t \ge r_f \end{cases} =$$



instead. It helps: