Control Theory (00350188)
lecture no. 2

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Outline

Reference signals in setpoint tracking problems

Reference profile: fastest realistic response and S-curves

Reference profile: fastest response under voltage constraints in DC motor

Anti-windup control

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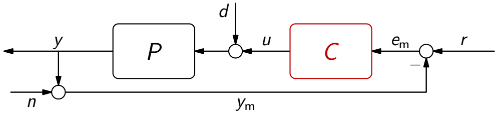
Anti-windup control

Step reference



By considering $r = y_{\mathrm{f}} \mathbb{1}$ we express the steady-state goal to

- reach the final setpoint $\lim_{t\to\infty} y(t) = y_{\mathrm{f}}$.

Step r is also used as a test signal to characterize quality of transients, e.g.

- overshoot
- raise time
- settling time

in controller design. This is convenient (analysis simplified / standardized). But

- does it make sense to use steps as *actual* reference signals ?
- can we do better via different r even if the final goal is a setpoint ?

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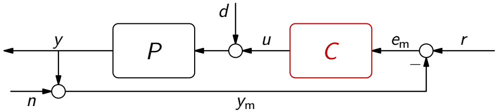

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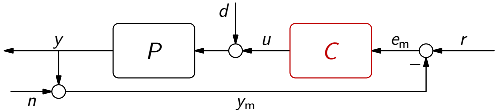

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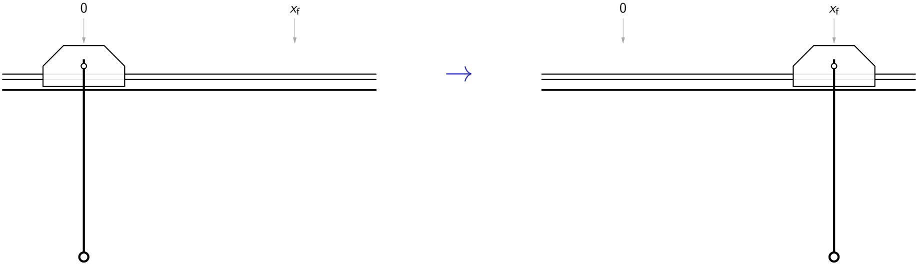
Example 1: moving cart with pendulum

Consider an undamped pendulum on a cart. The control input is the cart position x, the output is the pendulum angle θ. The linearized plant

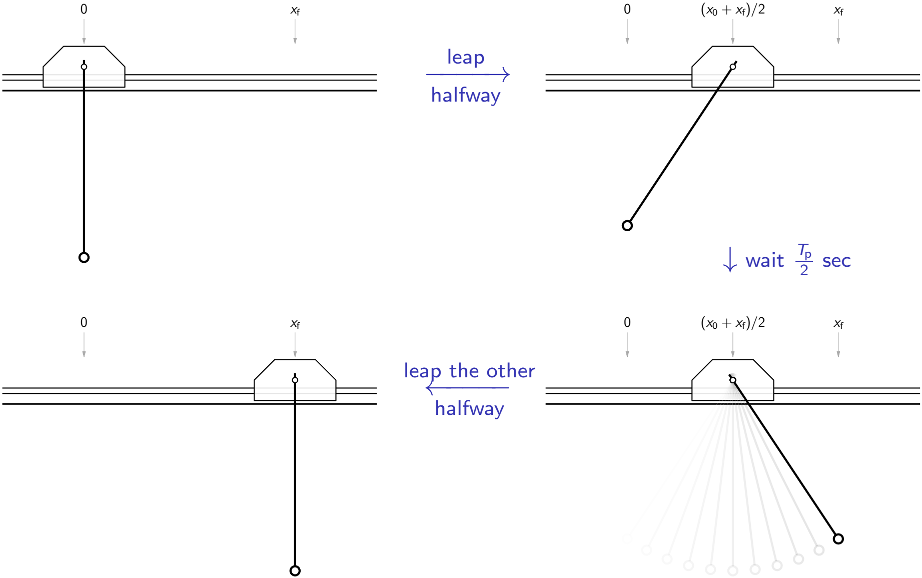
$$P(s) = \frac{s^2}{ls^2 + g},$$

where l is the pendulum length (so period $T_p := 2\pi\sqrt{l/g}$). Our goal is to

- move the cart *quickly* from $x = 0$ to x_f w/o oscillating the pendulum.



Example 1: posicast control (by Otto J. M. Smith)



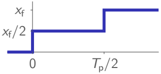
Example 1: moral

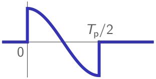
There is

- more than the (smoothened) step reference

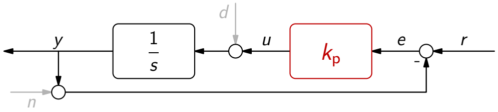
and

- transients can be improved by an elaborate choices of the command signal:

$x(t) =$ 

$\Longrightarrow$ $\theta(t) =$ 

Example 2



with

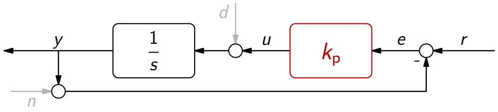
$$T(s) = \frac{k_p}{s + k_p} \quad \text{and} \quad T_c(s) = \frac{k_p s}{s + k_p}.$$

and the 5% settling time $t_s \approx 3\tau = 3/k_p$ independent of the setpoint y_f. If $r = y_f \mathbb{1}$, then, by linearity, both y and u are proportional to y_f.

meaning that

– large setpoint changes might cause actuator "overflow".

Example 2



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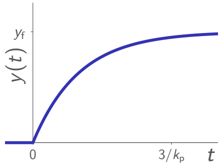

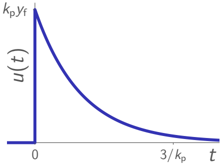

meaning that

– large setpoint changes might cause actuator "overflow".

Example 2: moral

When faces real-world limitations,

- linearity sucks

in the choice of the reference signal. Even the choice

$$r = T_{\mathrm{ref}} y_{\mathrm{f}} \mathbb{1}$$

for a low-pass T_{ref} that smoothens the reference signal won't resolve that.

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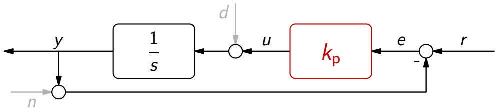
Anti-windup control

What's wrong here?



The problem is that the

- step reference is not realistic for inertial systems,

no inertial system can be expected to jump under a limited input. As such, the more successful we are in following such a command, the less affordable the price is.

For a problem to be realistic (resources are always limited),

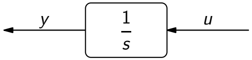
- t_s should depend on the setpoint change

and, in control terms, we need a

- nonlinear dependence of r on the setpoint.

Realistic settling time

Consider



under constraint $|u(t)| \leq u_{\max}$. Observe that

- t_s decreases as $|\dot{y}|$ increases
- $|\dot{y}(t)| = |u(t)|$
- y stops immediately as $u = 0$

Thus, $|\dot{y}(t)| \leq u_{\max}$ and the shortest t_s requires $|u(t)| = u_{\max}$ till $y(t) = y_f$. Hence,

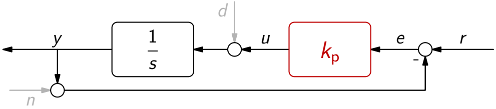
$$t_s \geq t_{s,\min} := |y_f|/u_{\max}.$$

This bound depends on the setpoint y_f (and on $u_{\max}$) and is attained via

$$u(t) = \begin{cases} \operatorname{sign}(y_f)u_{\max} & \text{if } t \leq t_{s,\min} \\ 0 & \text{if } t > t_{s,\min} \end{cases} \quad \text{and} \quad y(t) = \begin{cases} \operatorname{sign}(y_f)u_{\max}t & \text{if } t \leq t_{s,\min} \\ y_f & \text{if } t \geq t_{s,\min} \end{cases}$$

which are nonlinear functions of y_f.

Unity-feedback workaround

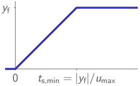


Let's pick r that yields

- the fastest system response under given physical constraints,

which in our case results in

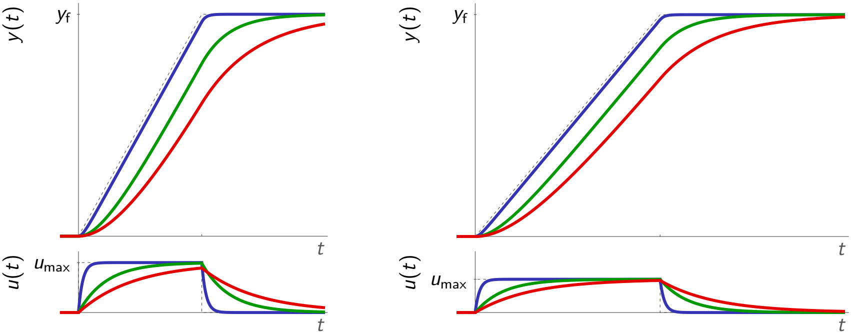
$$r(t) = \begin{cases} \operatorname{sign}(y_\mathrm{f})u_\mathrm{max}t & \text{if } t \le t_\mathrm{s,min} \\ y_\mathrm{f} & \text{if } t \ge t_\mathrm{s,min} \end{cases} =$$



instead of $r = y_\mathrm{f}\mathbb{1}$.

Unity-feedback: simulation results

For controller gains $k_p = 13$, $k_p = 2$, and $k_p = 1$ and two different u_{max}'s:



- r now agrees with the physics / limitations of the system
  hence, the resulting control signal is within the limits for all k_p
- tracking properties still depend on k_p (i.e. on closed-loop bandwidth)
  it's our job to pick agreeing k_p and r

Unity-feedback: simulation results (contd)

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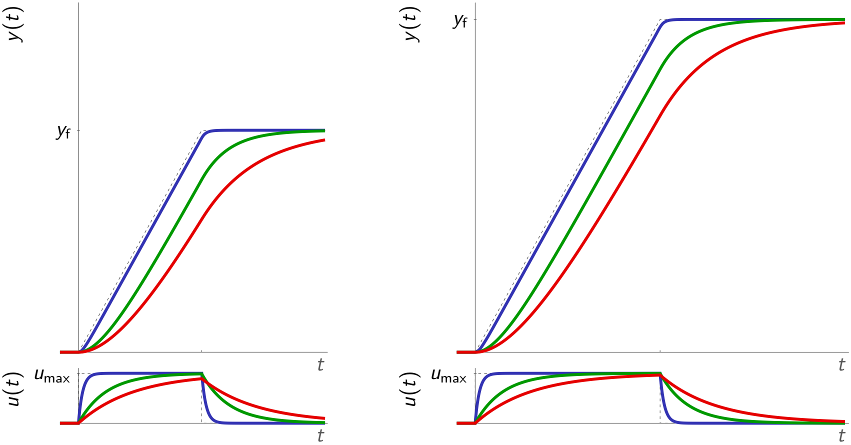
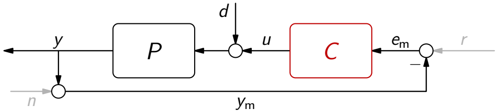


General considerations



Ideally, pick r that yields

- the fastest system response under given physical constraints.

But this might be rather knotty for

- more complex dynamics

  even for 2-order systems solution more complicated; no analytic solution in general

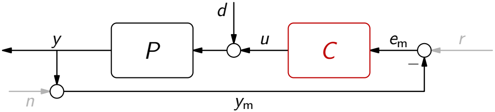
- more complex constraints

  might involve internal signals, like DC motor current, sensor limitations, et cetera

- nonzero initial conditions

  e.g. if a new setpoint arrives before the previous one was reached

Pragmatic alternative



Pick r that yields

– the fastest trajectory under given constraints on derivatives of r.

For example,

$$\begin{aligned} \text{minimize} \quad & t_f \\ \text{subject to} \quad & r(0) = 0, \quad r(t_f) = y_f, \quad \dot{r}(t_f) = 0, \quad \ddot{r}(t_f) = 0, \ldots \\ & |\dot{r}(t)| \leq v_{max} \\ & |\ddot{r}(t)| \leq a_{max} \\ & |\dddot{r}(t)| \leq j_{max} \end{aligned}$$

for given $v_{max} > 0$ (velocity), $a_{max} > 0$ (acceleration), and $j_{max} > 0$ (jerk), which indirectly reflect physical constraints, and a given setpoint y_f.

Example: constraints on velocity and acceleration

Problem:

$$\begin{array}{ll} \text{minimize} & t_\mathrm{f} \\ \text{subject to} & r(0) = 0, \quad r(t_\mathrm{f}) = y_\mathrm{f}, \quad \dot{r}(t_\mathrm{f}) = 0 \\ & |\dot{r}(t)| \le v_\mathrm{max}, \quad |\ddot{r}(t)| \le a_\mathrm{max} \end{array}$$

for given $v_\mathrm{max} > 0$, $a_\mathrm{max} > 0$, and y_f.

Complications (due to $a_\mathrm{max} < \infty$):

- maximal velocity cannot be achieved from the beginning
- r cannot be stopped immediately if its velocity is nonzero

Strategy:

1. start with maximal acceleration / stop with maximal deceleration
2. this might be sufficient (if y_f is so small that v_max is not reached)
3. if not, reset acceleration at $t = t_\mathrm{sw1}$, where $|\dot{r}(t_\mathrm{sw1})| = v_\mathrm{max}$ is satisfied, then start deceleration at $t = t_\mathrm{sw2}$, for which $r(t_\mathrm{sw2}) = y_\mathrm{f} - r(t_\mathrm{sw1})$

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Example: some calculations

1. Maximal acceleration (assume, for simplicity, that $y_f > 0$):

$$\ddot{r}(t) = a_{max} \implies \dot{r}(t) = a_{max} t \implies r(t) = a_{max} t^2/2.$$

Then $r(t) = y_f/2$ at $t_{sw} = \sqrt{y_f/a_{max}}$, so that

$$t_f = 2\sqrt{y_f/a_{max}} \quad \text{and} \quad \dot{r}(t_{sw}) = \sqrt{y_f\, a_{max}}.$$

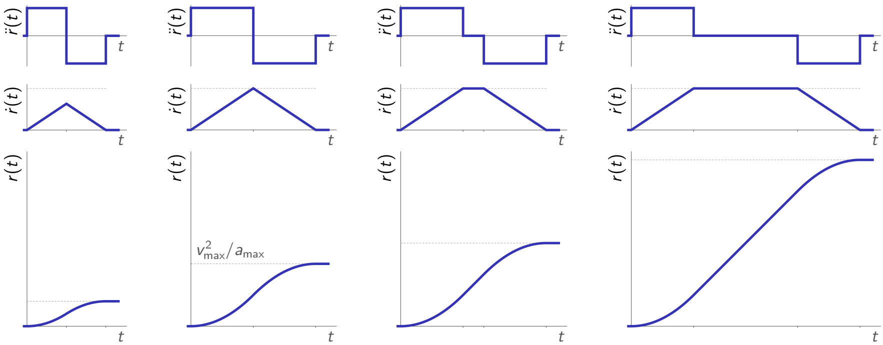
2. This strategy suffices iff $\sqrt{y_f\, a_{max}} \leq v_{max} \iff y_f \leq v_{max}^2/a_{max}$.

3. The first switch is at $\dot{r}(t_{sw1}) = v_{max}$, therefore $t_{sw1} = v_{max}/a_{max}$. At this moment $r(t_{sw1}) = v_{max}^2/(2a_{max}) < y_f/2$ and continues linearly, as

$$r(t) = v_{max}^2/(2a_{max}) + v_{max}(t - t_{sw1}) = v_{max} t - v_{max}^2/(2a_{max}).$$

The second switch happens at $r(t_{sw2}) = y_f - v_{max}^2/(2a_{max})$, from which $t_{sw2} = y_f/v_{max}$. Finally, because of symmetry $t_f = t_{sw1} + t_{sw2}$.

Reference trajectories (S-curve profiles)



with the settling time:

$$t_s = \begin{cases} 2\sqrt{|y_f|/a_{\max}} & \text{if } y_f \leq v_{\max}^2/a_{\max} \\ |y_f|/v_{\max} + v_{\max}/a_{\max} & \text{if } y_f \geq v_{\max}^2/a_{\max} \end{cases}$$

and switches at $t_{sw} = \sqrt{|y_f|/a_{\max}}$ or $t_{sw1} = v_{\max}/a_{\max}$ and $t_{sw2} = |y_f|/v_{\max}$.

Outline

Preliminaries: residues, simple poles case

Let $G(s)$ have a *simple* pole at $s = a$. The residue of G at $s = a$,

$$\operatorname{Res}(G(s), a) := \lim_{s \to a}(s - a)G(s).$$

If $\operatorname{Res}(G(s), a) = 0$, then the singularity at $s = a$ is removable.

If $G(s)$ is rational, proper, and has only simple poles, at $s = s_i$, then

$$G(s) = G(\infty) + \sum_{i=1}^{n} \frac{\operatorname{Res}(G(s), s_i)}{s - s_i}$$

(partial fraction expansion).

Preliminaries: residues, simple poles case (contd)

Example

$$G(s) = \frac{s^2}{ls^2 + g} \quad \Longrightarrow \quad G(s) = \frac{1}{l} + \frac{\mathrm{j}\sqrt{g/(4l^3)}}{s - \mathrm{j}\sqrt{g/l}} - \frac{\mathrm{j}\sqrt{g/(4l^3)}}{s + \mathrm{j}\sqrt{g/l}}$$

Example

The function

$$G(s) = \frac{1 - \alpha e^{-Ts}}{s}$$

has a single singularity at $s = 0$.

$$\operatorname{Res}(G(s), 0) = \lim_{s \to 0} sG(s) = \lim_{s \to 0} (1 - \alpha e^{-Ts}) = 1 - \alpha.$$

Two cases:

- $\alpha \neq 1$: $\operatorname{Res}(G(s), 0) \neq 0$ and the singularity is a pole
- $\alpha = 1$: $\operatorname{Res}(G(s), 0) = 0$ and the singularity is removable (not a pole)

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Preliminaries: a special complex function

Let

$$G(s) = \frac{N_0(s) + N_1(s)\mathrm{e}^{-\tau_1 s} + \cdots + N_n(s)\mathrm{e}^{-\tau_n s}}{D(s)} \qquad (*)$$

for polynomials $D(s)$ and $N_i(s)$ such that

- $\deg D(s) \geq \deg N_i(s)$ for all $i = 0, \ldots, n$,
- all roots s_i of $D(s)$ are simple,

and $0 < \tau_1 < \tau_2 < \cdots < \tau_n$. Expand, for $j = 0, \ldots, n$,

$$\frac{N_j(s)}{D(s)} = \beta_j + \sum_i \frac{\alpha_{ij}}{s - s_i}, \quad \alpha_{ij} = \operatorname{Res}\left(\frac{N_j(s)}{D(s)}, s_i\right) \text{ and } \beta_j = \lim_{s\to\infty} \frac{N_j(s)}{D(s)}.$$

Hence

$$G(s) = \beta(s) + \sum_i \frac{\alpha_i(s)}{s - s_i}, \quad \alpha_i(s) := \sum_{j=0}^{n} \alpha_{ij}\mathrm{e}^{-\tau_j s} \text{ and } \beta(s) := \sum_{j=0}^{n} \beta_j \mathrm{e}^{-\tau_j s}.$$

Note that $\alpha_i(s_i) = \operatorname{Res}(G(s), s_i)$.

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Preliminaries: impulse response of $(*)$

Let

$$G_i(s) := \frac{\alpha_i(s)}{s - s_i} = \sum_{j=0}^{n} \frac{\alpha_{ij}}{s - s_i} e^{-\tau_j s}.$$

Its inverse Laplace transform

$$g_i(t) = \sum_{j=0}^{n} \alpha_{ij} e^{s_i(t-\tau_j)} \mathbb{1}(t - \tau_j) = e^{s_i t} \sum_{j=0}^{n} \alpha_{ij} e^{-s_i \tau_j} \mathbb{1}(t - \tau_j)$$

If $t > \tau_n$, then $\mathbb{1}(t - \tau_j) = 1$ for all j and

$$g_i(t) = e^{s_i t} \sum_{j=0}^{n} \alpha_{ij} e^{-s_i \tau_j} = e^{s_i t} \alpha_i(s_i) \overset{\alpha_i(s_i)=0}{=} 0, \qquad \forall t > \tau_n.$$

Hence,

— $\alpha_i(s_i) = 0,\ \forall i \implies \operatorname{supp}(g) \subset [0, \tau_n]$ and G is BIBO stable.

Systems, whose impulse responses have support over finite intervals dubbed

— FIR (finite impulse response) systems.

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- $\alpha_i(s_i) = 0,\ \forall i \implies \mathrm{supp}(g) \subset [0, \tau_n]$ and G is BIBO stable.

Systems, whose impulse responses have support over finite intervals dubbed

- FIR (finite impulse response) systems.

Preliminaries: step response of $(*)$

The step response of G is

$$Y(s) = \frac{G(s)}{s} \iff y(t) = \int_0^t g(\theta)\mathrm{d}\theta.$$

If $\alpha_i(s_i) = 0$ for all i, then $\mathrm{supp}(g) \subset [0, \tau_n]$ and

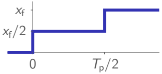
$$y(t) = \int_0^{\tau_n} g(\theta)\mathrm{d}\theta = \mathrm{const} = G(0), \quad \forall t > \tau_n.$$

In other words, the

– step response of FIR systems converges to steady state in finite time.

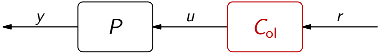
Remark: posicast control revisited

Control

$x(t) =$ 

$$\Longrightarrow \qquad X(s) = \frac{1 + e^{-sT_p/2}}{2}\,\frac{x_f}{s}$$

corresponds to the open-loop architecture (with $u = x$ and $y = \theta$)



under

- $P(s) = \dfrac{s^2}{ls^2 + g}$ and stable $C_{ol}(s) = \dfrac{1 + e^{-sT_p/2}}{2}$,
- step reference $r = x_f \mathbb{1}$

and the controlled system

$$P(s)C_{ol}(s) = \frac{0.5s^2 + 0.5s^2 e^{-sT_p/2}}{ls^2 + g}$$

is of form $(*)$ and has $\alpha_i(s_i) = 0$ for $i = 1, 2$, just because $C_{ol}(\pm j\sqrt{gl}\,) = 0$ (check it yourselves), so is FIR.

Remark: posicast control for dampened pendulum

Let

$$P(s) = \frac{s^2}{ls^2 + 2cs + g}, \quad \text{for } 0 \leq c < \sqrt{gl}$$

with poles at $-\sigma \pm \mathrm{j}\omega$ for $\sigma = c/l$ and $\omega = \sqrt{gl - c^2}/l = 2\pi/T_{\mathrm{p}}$. Choose

$$C_{\mathrm{ol}}(s) = \phi_0 + \phi_1 \mathrm{e}^{-\tau s}.$$

We shall require

- $C_{\mathrm{ol}}(0) = 1 = \phi_0 + \phi_1$ — $x = x_{\mathrm{f}}$ is steady state
- $C_{\mathrm{ol}}(-\sigma \pm \mathrm{j}\omega) = 0$ — posicast, i.e. FIR

Equivalent to

$$\begin{bmatrix} 1 & 1 \\ 1 & \mathrm{e}^{\tau\sigma}\cos(\tau\omega) \\ 0 & \mathrm{e}^{\tau\sigma}\sin(\tau\omega) \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(because $\mathrm{e}^{-\tau(-\sigma \pm \mathrm{j}\omega)} = \mathrm{e}^{\tau\sigma}(\sin(\tau\omega) \mp \mathrm{j}\cos(\tau\omega))$ in $\phi_0, \phi_1 \in \mathbb{R}$ and $\tau > 0$.

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Remark: posicast control for dampened pendulum (contd)

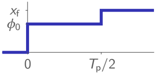
As $\phi_1 \neq 0$ (otherwise unsolvable), must have $\sin(\tau\omega) = 0$, with the shortest

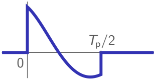
$$\tau = \frac{\pi}{\omega} \implies \begin{bmatrix} 1 & 1 \\ 1 & -\mathrm{e}^{\tau\sigma} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\tau\sigma} \\ 1 \end{bmatrix} \frac{1}{1 + \mathrm{e}^{\tau\sigma}}$$

($\phi_0 > 1/2$ if $c > 0$). Taking into account that $T_\mathrm{p} = 2\pi/\omega$, we end up with

$$C_\mathrm{ol}(s) = \frac{\mathrm{e}^{0.5 T_\mathrm{p} c/l} + \mathrm{e}^{-0.5 T_\mathrm{p} s}}{1 + \mathrm{e}^{0.5 T_\mathrm{p} c/l}}.$$

The resulting

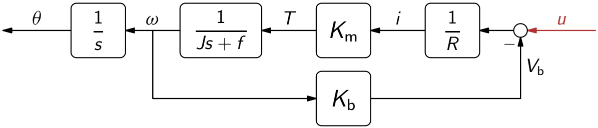
$x(t) =$ 

$\implies$ $\theta(t) =$ 

also finish the move in $T_\mathrm{p}/2$, but

– not posicast, in the sense that $\dot{\theta}(t)|_{t \uparrow T_\mathrm{p}/2} \neq 0$, whenever $c \neq 0$.

Fastest shaft angle change under voltage constraints



Consider the task of turning the shaft of a DC motor resting at $\theta(0) = \theta_0$ to a new angular position, say $\theta_f \neq \theta_0$, and resting there. We may need to

- do that as quick as possible under physical constraints.

A possible constraint[1] is the

- input voltage amplitude, $|u(t)| \leq u_{\max}$ for some $u_{\max} > 0$.

Our goal is to generate u that may then be a good choice for the reference trajectory r.

[1] The armature current amplitude is another, perhaps even more practical, possibility.

Mathematical formulation

Let θ satisfy

$$RJ\ddot{\theta}(t) + (Rf + K_\mathrm{m}K_\mathrm{b})\dot{\theta}(t) = K_\mathrm{m}u(t) \quad \Longleftrightarrow \quad \tau\ddot{\theta}(t) + \dot{\theta}(t) = ku(t)$$

for $\tau := RJ/(Rf + K_\mathrm{m}K_\mathrm{b})$ and $k := K_\mathrm{m}/(Rf + K_\mathrm{m}K_\mathrm{b})$,

$$\begin{array}{ll} \text{minimize} & t_\mathrm{f} \\ \text{subject to} & \theta(0) = \theta_0, \quad \dot{\theta}(0) = 0, \quad \theta(t_\mathrm{f}) = \theta_\mathrm{f}, \quad \dot{\theta}(t_\mathrm{f}) = 0 \\ & |u(t)| \le u_\mathrm{max} \end{array}$$

for given θ_0, θ_f, and $u_\mathrm{max} > 0$. This problem depends on system dynamics.

Note that the model in the Laplace variable domain,

$$\Theta(s) = \frac{\theta_0}{s} + \frac{k}{s(\tau s + 1)}U(s),$$

is affected by the initial condition.

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for given θ_0, θ_f, and $u_\mathrm{max} > 0$. This problem depends on system dynamics. Note that the model in the Laplace variable domain,

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is affected by the initial condition.

Time-optimal control

The studied problem is a special case of the time-optimal control problems, whose theory goes beyond the scope of this course. Outcomes of the theory relevant for the discussion below are:

- optimal $u(t)$ in $0 < t < t_{\mathrm{f}}$ takes values only in the set $\{-u_{\max}, u_{\max}\}$
  (such control strategy is known as bang-bang control)
- there is a finite number of switches $u_{\max} \rightleftharpoons -u_{\max}$ for any finite t_{f}
- if the plant has only real poles, say n, then the number of switches in $(0, t_{\mathrm{f}})$ is at most $n-1$

Applying to our problem,

$$u(t) = \begin{cases} u_1 & \text{if } t \in (0, t_{\mathrm{sw}}) \\ -u_1 & \text{if } t \in (t_{\mathrm{sw}}, t_{\mathrm{f}}) \\ 0 & \text{if } t \in (t_{\mathrm{f}}, \infty) \end{cases} =$$

for $|u_1| = u_{\max}$ and some $0 < t_{\mathrm{sw}} < t_{\mathrm{f}}$ to be determined.

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Applying to our problem[2],

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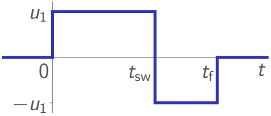


for $|u_1| = u_{\max}$ and some $0 < t_{\mathrm{sw}} < t_{\mathrm{f}}$ to be determined.

[2] Mind that $u(t) = 0$ whenever $t \notin [0, t_{\mathrm{f}}]$ because of an integrator in the plant.

Solution logic

Thus, $u(t) = u_1\big(\mathbb{1}(t) - 2\mathbb{1}(t - t_{\mathrm{sw}}) + \mathbb{1}(t - t_{\mathrm{f}})\big)$, or

$$U(s) = u_1 \frac{1 - 2\mathrm{e}^{-st_{\mathrm{sw}}} + \mathrm{e}^{-st_{\mathrm{f}}}}{s},$$

and

$$\Theta(s) = \frac{\theta_0}{s} + \frac{ku_1(1 - 2\mathrm{e}^{-st_{\mathrm{sw}}} + \mathrm{e}^{-st_{\mathrm{f}}})}{s^2(\tau s + 1)} = \overbrace{\left(\theta_0 + \frac{ku_1(1 - 2\mathrm{e}^{-st_{\mathrm{sw}}} + \mathrm{e}^{-st_{\mathrm{f}}})}{s(\tau s + 1)}\right)}^{G_\theta(s)}\frac{1}{s}$$

Our goal is to

– determine $\mathrm{sign}(u_1)$, t_{sw}, and $t_{\mathrm{f}} > t_{\mathrm{sw}}$

such that $G_\theta(s)$ is FIR and $G_\theta(0) = \theta_{\mathrm{f}}$. This is equivalent[3] to

1. $\lim_{s\to 0} G_\theta(s) = \theta_{\mathrm{f}}$
2. $\mathrm{Res}(G_\theta(s), -1/\tau) = 0$

[3] Mind that the singularity of $G_\theta(s)$ at $s = 0$ is always removable, by construction.

Solution details

1. Condition $\lim_{s\to 0} G_\theta(s) = \theta_\mathrm{f}$ reads

$$\theta_\mathrm{f} = \theta_0 + \lim_{s\to 0} \frac{ku_1(1 - 2\mathrm{e}^{-st_\mathrm{sw}} + \mathrm{e}^{-st_\mathrm{f}})}{s(\tau s + 1)} = \theta_0 + ku_1(2t_\mathrm{sw} - t_\mathrm{f}).$$

   Hence,

$$ku_1(2t_\mathrm{sw} - t_\mathrm{f}) = \theta_\mathrm{f} - \theta_0.$$

2. Condition $\mathrm{Res}(G_\theta(s), -1/\tau) = 0$ reads

$$0 = \lim_{s\to -1/\tau}\left(s + \frac{1}{\tau}\right)G_\theta(s) = \lim_{s\to -1/\tau} \frac{ku_1(1 - 2\mathrm{e}^{-st_\mathrm{sw}} + \mathrm{e}^{-st_\mathrm{f}})}{\tau s}$$
$$= -ku_1(1 - 2\mathrm{e}^{t_\mathrm{sw}/\tau} + \mathrm{e}^{t_\mathrm{f}/\tau}).$$

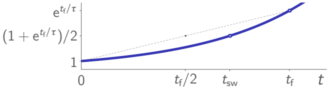
   Hence,

$$\mathrm{e}^{t_\mathrm{sw}/\tau} = \frac{1 + \mathrm{e}^{t_\mathrm{f}/\tau}}{2}.$$

Solution details (contd)

The equality

$$
\mathrm{e}^{t_{\mathrm{sw}}/\tau} = \frac{1 + \mathrm{e}^{t_{\mathrm{f}}/\tau}}{2} \quad :
$$



implies that $t_{\mathrm{sw}} > t_{\mathrm{f}}/2$, because

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{t/\tau} = \frac{\mathrm{e}^{t/\tau}}{\tau} > 0 \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{e}^{t/\tau} = \frac{\mathrm{e}^{t/\tau}}{\tau^2} > 0
$$

for all t (meaning that $\mathrm{e}^{t/\tau}$ is increasing and strictly convex). But then

$$
(2t_{\mathrm{sw}} > t_{\mathrm{f}}) \wedge (k u_1 (2t_{\mathrm{sw}} - t_{\mathrm{f}}) = \theta_{\mathrm{f}} - \theta_0) \implies \operatorname{sign}(u_1) = \operatorname{sign}(\theta_{\mathrm{f}} - \theta_0)
$$

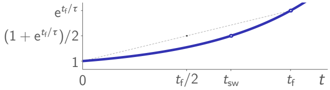
and

$$
\frac{\theta_{\mathrm{f}} - \theta_0}{u_1} = \frac{|\theta_{\mathrm{f}} - \theta_0|}{u_{\max}}.
$$

Solution details (contd)

The equality

$$e^{t_{sw}/\tau} = \frac{1 + e^{t_f/\tau}}{2} \quad :$$



implies that $t_{sw} > t_f/2$, because

$$\frac{d}{dt} e^{t/\tau} = \frac{e^{t/\tau}}{\tau} > 0 \quad \text{and} \quad \frac{d^2}{dt^2} e^{t/\tau} = \frac{e^{t/\tau}}{\tau^2} > 0$$

for all t (meaning that $e^{t/\tau}$ is increasing and strictly convex). But then

$$(2t_{sw} > t_f) \land (ku_1(2t_{sw} - t_f) = \theta_f - \theta_0) \implies \text{sign}(u_1) = \text{sign}(\theta_f - \theta_0)$$

and

$$\frac{\theta_f - \theta_0}{u_1} = \frac{|\theta_f - \theta_0|}{u_{max}}.$$

Solution details (contd)

Thus, we end up with the following two equations for $t_{\mathsf{sw}} > 0$ and $t_{\mathsf{f}} > t_{\mathsf{sw}}$:

$$2t_{\mathsf{sw}} - t_{\mathsf{f}} = \frac{|\theta_{\mathsf{f}} - \theta_0|}{k u_{\max}} \quad \text{and} \quad 2\mathrm{e}^{t_{\mathsf{sw}}/\tau} = 1 + \mathrm{e}^{t_{\mathsf{f}}/\tau}.$$

Hence, $t_{\mathsf{f}} = 2t_{\mathsf{sw}} - |\theta_{\mathsf{f}} - \theta_0|/(k u_{\max})$ and

$$\mathrm{e}^{-|\theta_{\mathsf{f}} - \theta_0|/(\tau k u_{\max})} (\mathrm{e}^{t_{\mathsf{sw}}/\tau})^2 - 2\mathrm{e}^{t_{\mathsf{sw}}/\tau} + 1 = 0.$$

Solving this quadratic equation in $\mathrm{e}^{t_{\mathsf{sw}}/\tau}$ yields (take "+" to have $t_{\mathsf{sw}} < t_{\mathsf{f}}$)

$$t_{\mathsf{sw}} = \frac{|\theta_{\mathsf{f}} - \theta_0|}{k u_{\max}} + \tau \ln\left(1 + \sqrt{1 - \mathrm{e}^{-|\theta_{\mathsf{f}} - \theta_0|/(\tau k u_{\max})}}\right)$$

and

$$t_{\mathsf{f}} = \frac{|\theta_{\mathsf{f}} - \theta_0|}{k u_{\max}} + 2\tau \ln\left(1 + \sqrt{1 - \mathrm{e}^{-|\theta_{\mathsf{f}} - \theta_0|/(\tau k u_{\max})}}\right).$$

Both are increasing functions of $|\theta_{\mathsf{f}} - \theta_0|$ and τ and decreasing of $k u_{\max}$.

Solution details (contd)

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Both are increasing functions of $|\theta_{\mathrm{f}} - \theta_0|$ and τ and decreasing of $ku_{\max}$.

The fastest $\theta(t)$

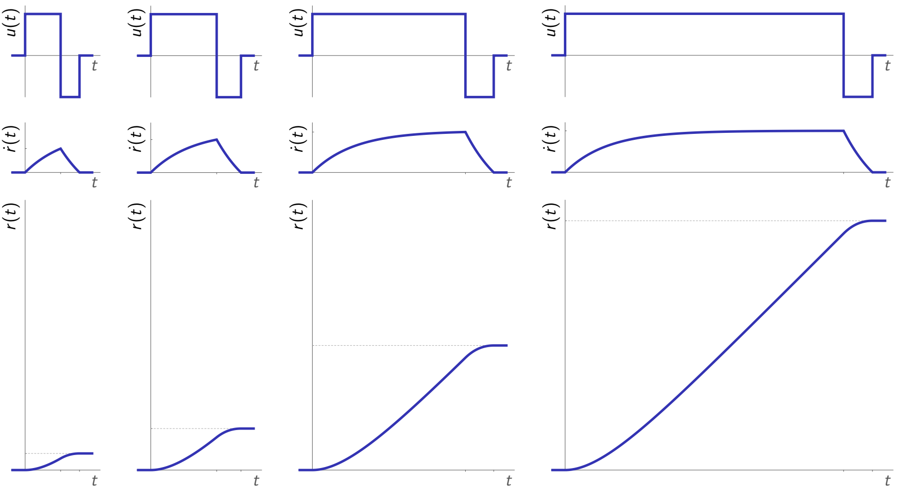
Taking the inverse Laplace transform of $\Theta(s)$, we finally get

$$\theta(t) = \begin{cases} \theta_0 + \big(t - (1 - \mathrm{e}^{-t/\tau})\tau\big)ku_1 & \text{if } t \in [0, t_{\mathrm{sw}}] \\ \theta_0 + \big(2t_{\mathrm{sw}} + \tau - t - \mathrm{e}^{-t/\tau}(2\mathrm{e}^{t_{\mathrm{sw}}/\tau} - 1)\tau\big)ku_1 & \text{if } t \in [t_{\mathrm{sw}}, t_{\mathrm{f}}] \\ \theta_{\mathrm{f}} & \text{if } t \in [t_{\mathrm{f}}, \infty) \end{cases}$$

where $u_1 = \mathrm{sign}(\theta_{\mathrm{f}} - \theta_0)u_{\max}$. The corresponding angular velocity

$$\omega(t) = \begin{cases} (1 - \mathrm{e}^{-t/\tau})ku_1 & \text{if } t \in [0, t_{\mathrm{sw}}] \\ \big(\mathrm{e}^{-t/\tau}(2\mathrm{e}^{t_{\mathrm{sw}}/\tau} - 1) - 1\big)ku_1 & \text{if } t \in [t_{\mathrm{sw}}, t_{\mathrm{f}}] \\ 0 & \text{if } t \in [t_{\mathrm{f}}, \infty) \end{cases}$$

Resulting reference trajectories



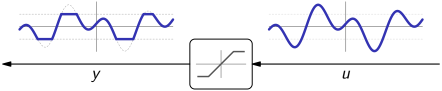
are reminiscent of S-curves, but are not symmetric (stopping is cheaper).

Outline

Saturation

It is a system $u \mapsto y$, which we denote $\text{sat}_{[a,b]}$, such that

$$y(t) = \begin{cases} a & \text{if } u(t) < a \\ u(t) & \text{if } a \le u(t) \le b \\ b & \text{if } u(t) > b \end{cases}$$

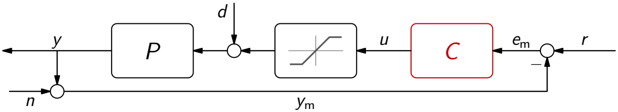


for given $a < b$. We use the short notation $\text{sat}_a := \text{sat}_{[-a,a]}$ for some $a > 0$. Think of a gas pedal in cars, water tap, integer overflow in computers, etc.

Saturation element is a nonlinear system (no superposition). Indeed,

$$\text{sat}_1(2 \times 0.6 \sin t) \neq 2 \times \text{sat}_1(0.6 \sin t) = 1.2 \sin t.$$

Saturation in feedback loop



All actuators saturate. Indeed,

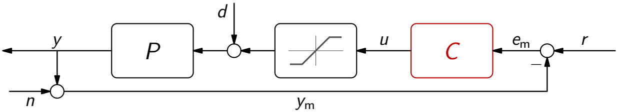
- force, torque,
- voltage, current,
- flow rate,
- ...

are ultimately limited. Some sensors saturate as well. We therefore must

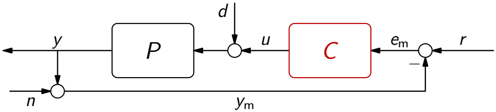
- respect the presence of (nonlinear) saturation elements

in any feedback loop.

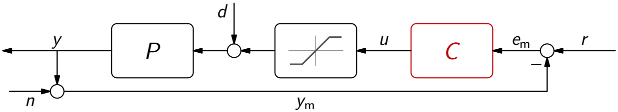
Saturation in feedback loop (contd)



– if u does not saturate, it behaves as standard linear closed-loop system:



Saturation in feedback loop (contd)



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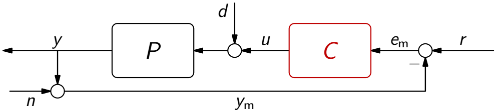
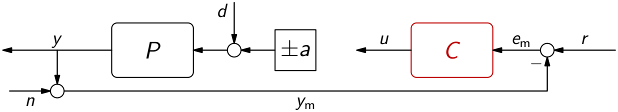
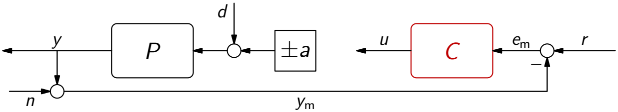


- if u saturates, it behaves as open-loop system:



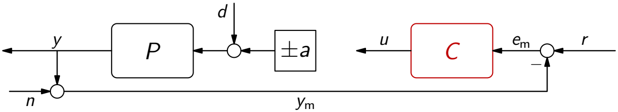
Saturation in feedback loop (contd)



This doesn't help in general, yet it is

– especially problematic when either $P(s)$ or $C(s)$ is unstable.

Saturation in feedback loop (contd)



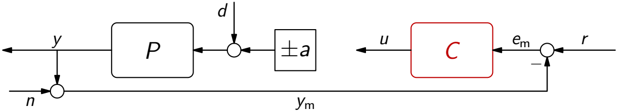
This doesn't help in general, yet it is

– especially problematic when either $P(s)$ or $C(s)$ is unstable.

What can be done:

– When plant is unstable, there is nothing we can do.

Saturation in feedback loop (contd)



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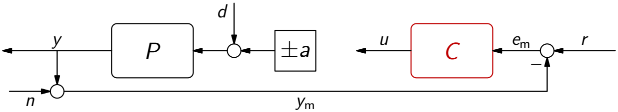

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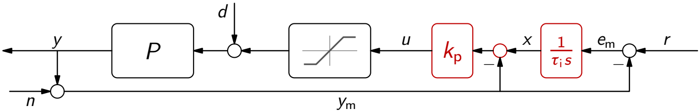
[4]E.g. the plant is not strongly stabilizable, an integral action is required, et cetera.

PI controllers and saturation



PI controller transforms y_m and r to control signal u according to

$$U(s) = \frac{k_p}{\tau_i s}\big(R(s) - Y_m(s)\big) - k_p Y_m(s) \quad \text{or} \quad \begin{cases} \dot{x}(t) = \frac{1}{\tau_i}\big(r(t) - y_m(t)\big) \\ u(t) = k_p\big(x(t) - y_m(t)\big) \end{cases}$$

While $u(t)$ saturates,

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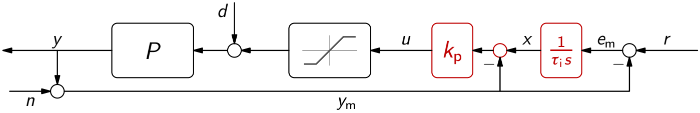

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[5] This is how this controller is implemented.

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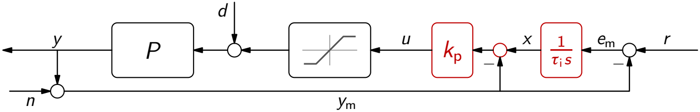

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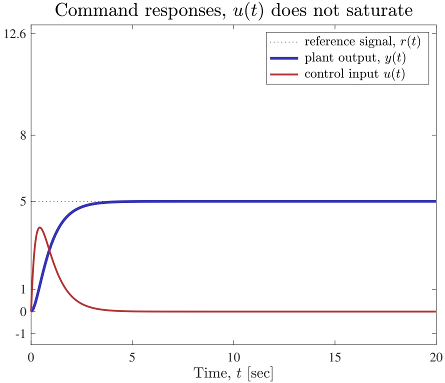
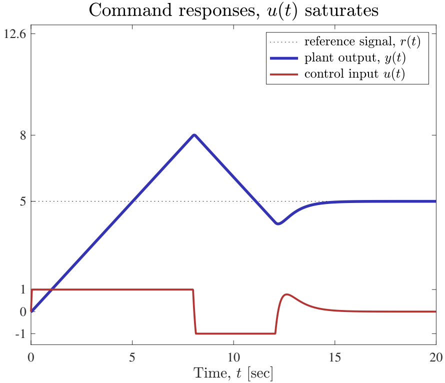
Consider control system with $P(s) = 1/s$ and $C(s) = 5(1 + 1/s)$. We have:





State variable becomes very large by the time error approaches 0, hence $\implies$ y continues to grow until $x(t)$ becomes smaller than $y(t)$ (remember, $u = k_p(x - y)$ and the direction of y equals the sign of $u = \dot{y}$).

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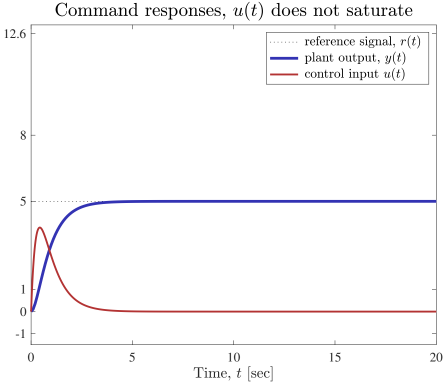

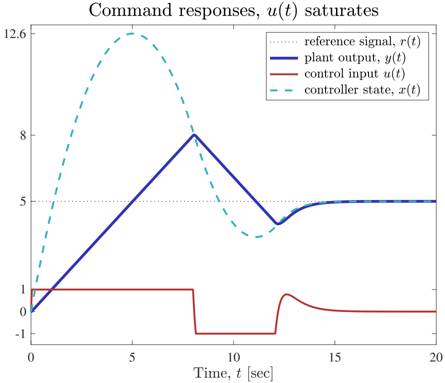

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Integrator windup

The effect of

- significant grow of the integrator state during actuator saturation

is called the integrator windup.

Arguably, most remedies for windup effect are based on

- preventing integrator state from unstable updating once u saturates.

Possible heuristics (sometimes equivalent):

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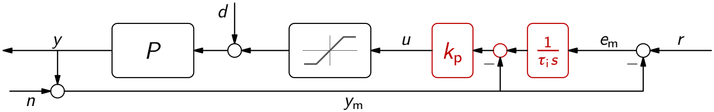
Anti-windup scheme with internal feedback



This scheme (τ_t is called the tracking time constant) works as follows:

- if u does not saturate, then $\mathrm{sat}_a(u) - u = 0$ and it is a standard PI;
- if u saturates, then the controller becomes stable:

$$U(s) = k_p\left(\frac{1}{\tau_i s}\left(R(s) - Y_m(s) \pm \frac{a}{\tau_t} - \frac{1}{\tau_t}U(s)\right) - Y_m(s)\right),$$

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$$\frac{\tau_t \tau_i s + k_p}{\tau_t \tau_i s}\,U(s) = \frac{k_p}{\tau_i s}\left(R(s) - (\tau_i s + 1)\,Y_m(s) \pm \frac{a}{\tau_t}\right)$$

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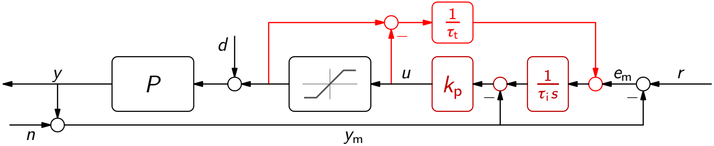

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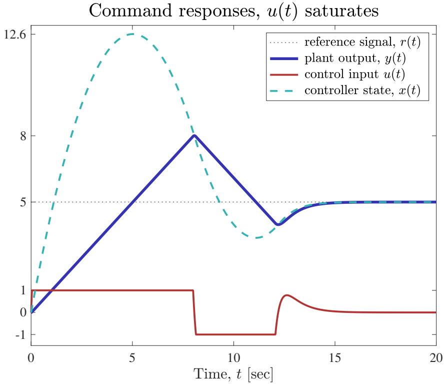
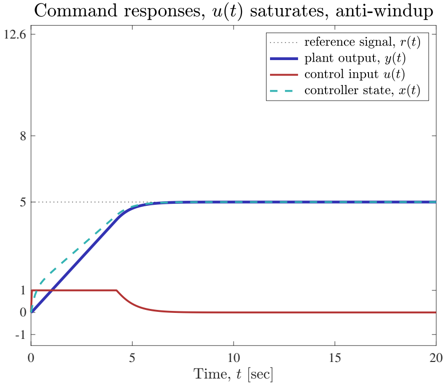
PI controllers and saturation: example (contd)

Internal feedback really helps (here $\tau_t = 1$):





Another anti-windup solution: saturation-aware r

In many situations we may

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so no need in smart solutions to problems one shouldn't have gotten into in the first place.

Example:

With $P(s) = \frac{1}{s}$ and $|u(t)| \le 1$ we have no chance to raise faster than $y(t) = t$ anyway. It may make sense to pick

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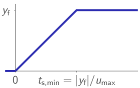

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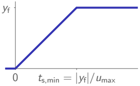

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