Control Theory (00350188)
lecture no. 1

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General info

- Course site: http://leo.technion.ac.il/Courses/CT/
- Credit points: 3.5
- Prerequisite: Introduction to Control (00340040), **a must**
- Grading policy:
  2 midterm projects (tokef): 20% each (provided exam is passed)
  Final exam ("closed"): 60% (or 100% is the grade is < 55)
- Passing policy:
  minimum passing grade is 55
  only those who pass both projects are eligible to take the final exam

Syllabus

1. Advanced single loop design
   - 1.1 More on loop-shaping
   - 1.2 More on on dead-time systems
   - 1.3 More on pole placement (Sylvester matrix etc.)
   - 1.4 Industrial control (saturation & anti-windup, reference signal generation)
   - 1.5 Robustness of control systems
2. Introduction to state-space methods
   - 2.1 Structural properties (controllability, observability, etc)
   - 2.2 State feedback control
   - 2.3 State observers
   - 2.4 Observer-based output feedback
   - 2.5 Introduction to optimization-based methods (LQR, Kalman filter, LQG)
3. Introduction to sampled-data systems
   - 3.1 Digital redesign of analog controllers
   - 3.2 Digital design

Outline

Loop-shaping tools

M- and N-circles

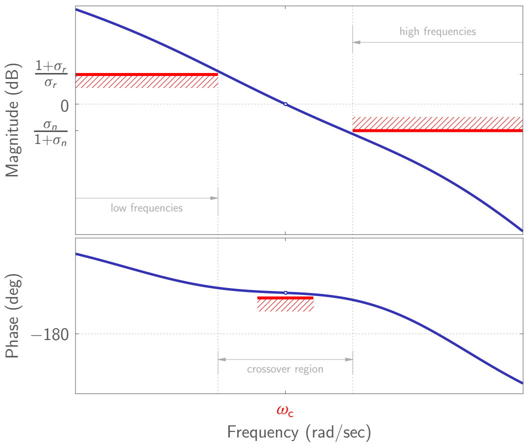
Nichols chart

Design example: the use of Nichols charts

Bode's gain-phase relation

Philosophical remark: Bode's sensitivity integral

Typical operations with $L(j\omega)$



Typical course of action:

- choose crossover, ω_c
- shape high-freq. roll-off (means: low-pass filter)
- set required crossover, ω_c (means: proportional controller)
- shape phase around ω_c (means: lead controller)
- shape low-frequency gain (means: lag controller)

by cascade adjustments of $C(s)$.

But

- one should not be religious about that,

the steps may be skipped, reordered, or altered, depending on the situation.

Low-pass filters: Butterworth

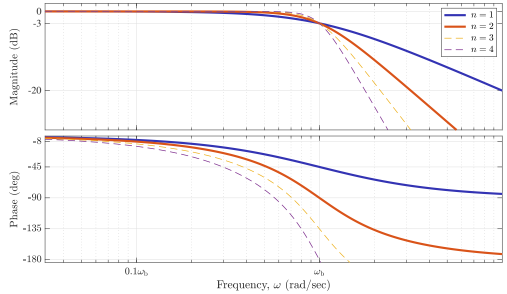
The n-order Butterworth filter[1] with bandwidth ω_b is the stable t.f. such that

$$|F_{b,n}(j\omega)|^2 = \frac{1}{1+(\omega/\omega_b)^{2n}},$$

like

$$F_{b,1}(s) = \frac{\omega_b}{s+\omega_b} \quad \text{or} \quad F_{b,2}(s) = \frac{\omega_b^2}{s^2+\sqrt{2}\omega_b s+\omega_b^2}$$

with

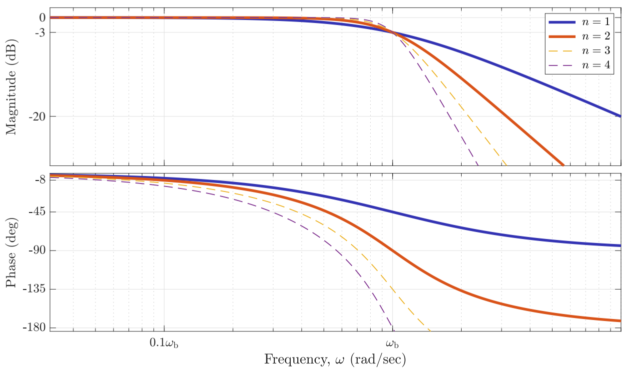


[1] MATLAB: `[num,den] = butter(n,wb,'s'); Fb = tf(num,den);`

Low-pass filter: usage

Main problem is that

- the phase lags before the magnitude starts to decay:



The rule of thumb:

- use $\omega_b = 10\omega_c$ (decade above the intended crossover),

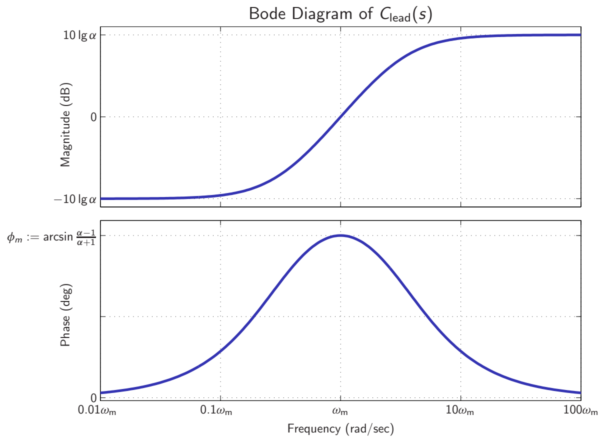
with $\angle F_{b,1}(j\omega_c) \approx -5.7°$, $\angle F_{b,2}(j\omega_c) \approx -8.1°$, $\angle F_{b,3}(j\omega_c) \approx -11.5°$, ...

1-order lead

General form:

$$C_{lead}(s) = \frac{\sqrt{\alpha}\, s+\omega_m}{s+\sqrt{\alpha}\omega_m}, \qquad \text{with } \alpha = \frac{1+\sin\phi_m}{1-\sin\phi_m},$$

where $\phi_m \in (0, 90°)$ is the maximal phase lead (occurs at $\omega = \omega_m$).

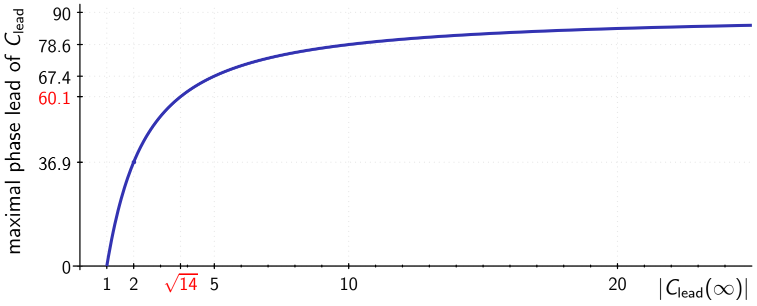


1-order lead: cost of phase lead

Phase lead is expensive, it leads to

- the decrease of the low-frequency gain and
- the increase of the high-frequency gain

of the controller (both by the factor $\sqrt{\alpha}$). Quantitatively,



whose slope decreases. A rule of thumb is that

- the phase lead above 60° might be too expensive.

2-order lead

General form:

$$C_{\text{lead2}}(s) = \frac{\alpha\, s^2 + 2\zeta\sqrt{\alpha}\omega_{\text{m}} s + \omega_{\text{m}}^2}{s^2 + 2\zeta\sqrt{\alpha}\omega_{\text{m}} s + \alpha\omega_{\text{m}}^2}, \qquad \text{with } \zeta \in \Big[\frac{1}{\sqrt{2}}, \sqrt{2}\Big],$$

and

$$\alpha = 1 + 2\zeta\Big(\zeta + \sqrt{\zeta^2 + \cot^2\tfrac{\phi_m}{2}}\Big)\tan^2\tfrac{\phi_m}{2}$$

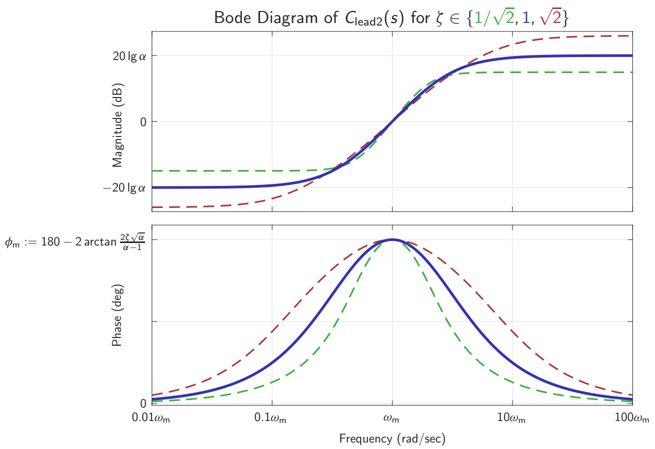
where $\phi_m \in (0, 180°)$ is the maximal phase lead (occurs at $\omega = \omega_{\text{m}}$). Here

- the case $\zeta = 1$ corresponds to $C_{\text{lead2}} = C_{\text{lead}}^2$,
- if $\zeta < 1/\sqrt{2}$, then $|C_{\text{lead2}}(j\omega)|$ is not monotonic, so might be trickier,
- if $\zeta > \sqrt{2}$, then it might be that $\angle C_{\text{lead2}}(j\omega) > \phi_m$ for some $\omega \neq \omega_{\text{m}}$.

2-order lead (contd)

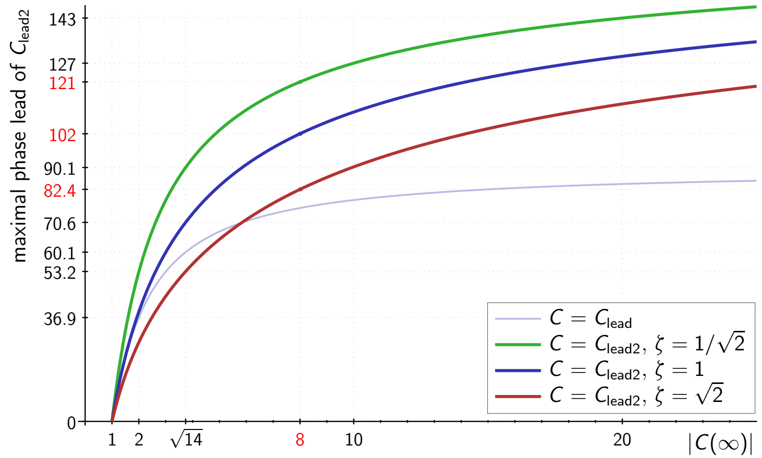
As ζ increases for the same ϕ_m,

- phase lead becomes wider
- α increases



2-order lead: cost of phase lead

Quantitatively,



A rule of thumb is that

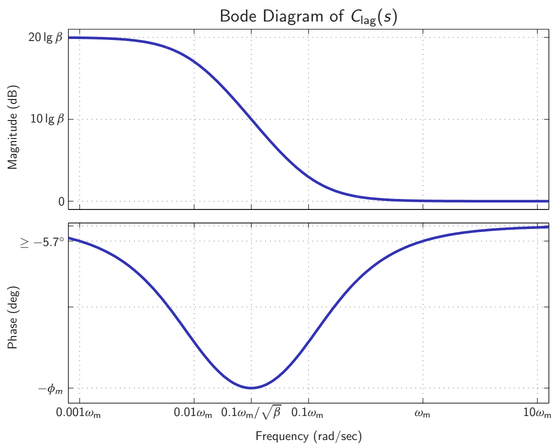
- the phase lead above 120° might be too expensive.

Lag

General form:

$$C_{\text{lag}}(s) = \frac{10s + \omega_m}{10s + \omega_m/\beta}, \quad \text{with } \beta > 1,$$

where the phase lag at $\omega = \omega_m$ is at most $5.7°$.



Outline

Loop-shaping tools

M- and *N*-circles

Nichols chart

Design example: the use of Nichols charts

Bode's gain-phase relation

Philosophical remark: Bode's sensitivity integral

M-circles

M-circles are contours of constant closed-loop magnitude on Nyquist plane.

Let $L(j\omega) = x + jy$. Then $T(j\omega) = \frac{x+jy}{1+x+jy}$. Hence,

$$|T(j\omega)|^2 = M^2 \iff M^2(1+x)^2 + M^2y^2 = x^2 + y^2$$
$$\iff (1 - M^2)x^2 - 2M^2x + (1 - M^2)y^2 = M^2.$$

Then two cases are possible:

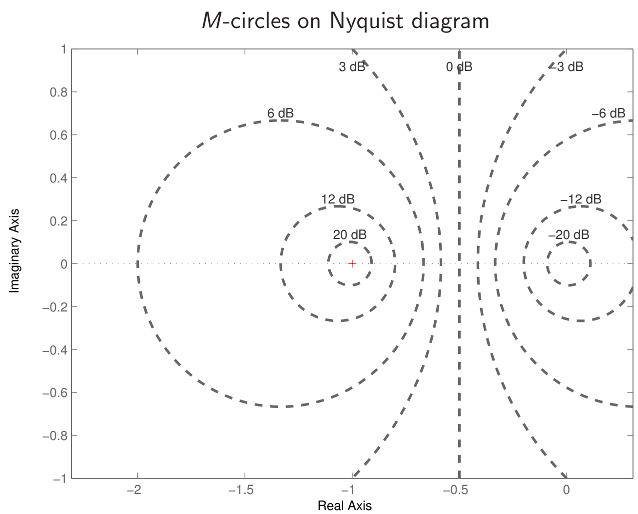
$M = 1$ then $x = -\frac{1}{2}$ (vertical line)

$M \neq 1$ then $(1 - M^2)\left(x^2 - 2\frac{M^2}{1-M^2}x \pm \frac{M^4}{(1-M^2)^2} + y^2\right) = M^2$, so we get:

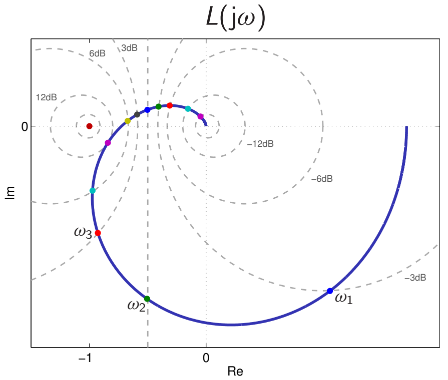
$$\left(x - \frac{M^2}{1-M^2}\right)^2 + y^2 = \left(\frac{M}{1-M^2}\right)^2$$

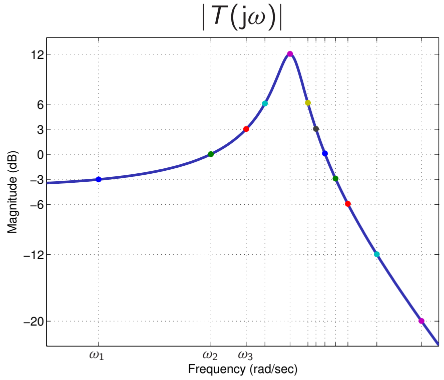
(circle centered at $\frac{M^2}{1-M^2}$ with radius $\frac{M}{|1-M^2|}$)

M-circles (contd)



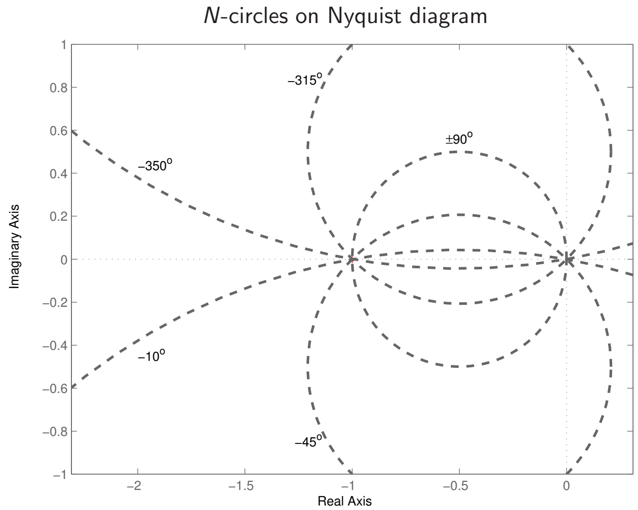
M-circles: how to read





N-circles

N-circles are contours of constant closed-loop phase on Nyquist plane:



Outline

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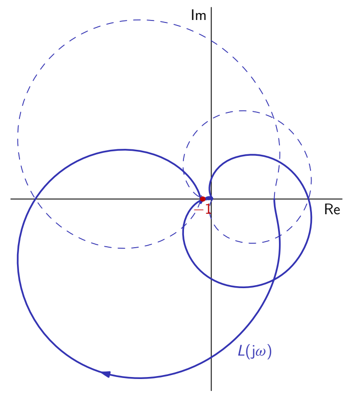
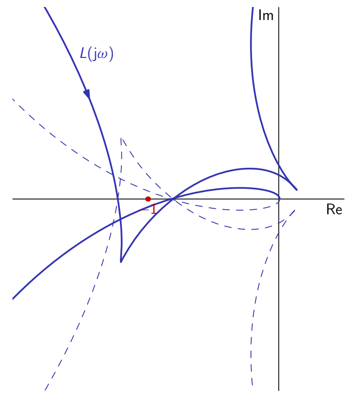
Nichols chart: motivation

Let $L(s) = \frac{12960000(s+5)(s^2+0.8s+16)(s^2+1.52s+1444)e^{-0.075s}}{(s+10)(s+100)(s^2+3s+9)(s^2+0.48s+9)(s^2+0.8s+1600)^2}$. Its Nyquist plot



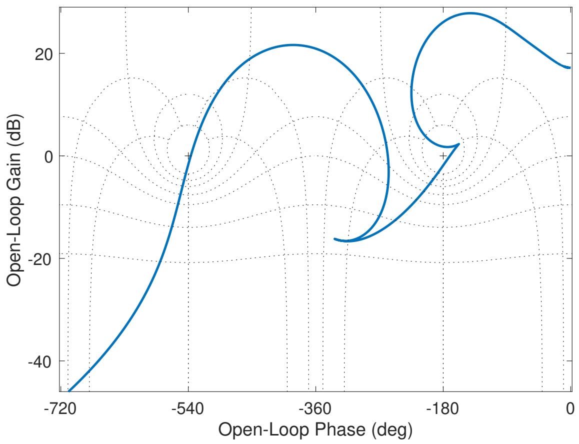


Pitfalls of the Nyquist plot:

- becomes messy for systems with multiple crossover frequencies
- crossover region is imperceptible for systems with large resonant peaks
- lacks system composition (superposition) properties of Bode diagrams

Remedy: Nichols chart

Nichols chart of transfer function $L(s)$ is plot of $|L(j\omega)|$ (in dB) vs. $\angle L(j\omega)$ (in degrees) as frequency ω changes from 0 to ∞.



Nichols chart: advantages

Since phase scale is linear rather than polar,

– Nichols chart is typically cleaner than Nyquist plot

especially for systems with large phase lags, like time-delay systems.

As magnitude scale is in dB, regions with large magnitude don't dominate, hence

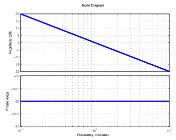
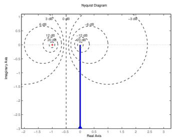
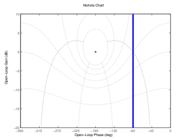
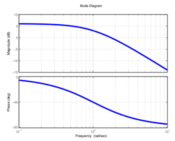
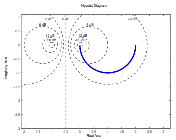
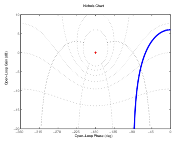
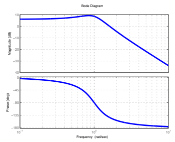
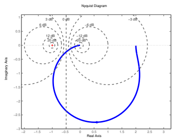
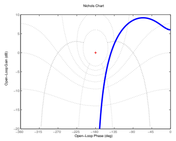
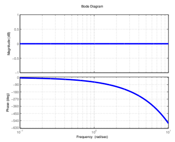
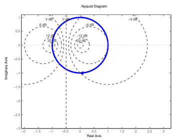
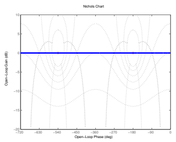
– the crossover region is more visible.

Also the consequence of the logarithmic scale of $|L(j\omega)|$ is that

– multiplication of systems results in superposition

on Nichols chart, almost as easy as on the Bode diagrams.

Nichols charts of elementary systems

$L(s)$	Bode	Nyquist	Nichols
$\frac{k}{s}$			
$\frac{k}{\tau s+1}$			
$\frac{k\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$			
e^{-sh}			

Nyquist criterion on Nichols chart

The same idea as with the Nyquist plot, we should

– count encirclements of the critical point by the frequency-response plot.

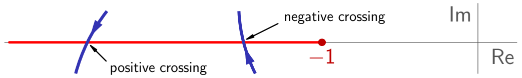
This procedure might be less tangible with the Nichols charts as

– the critical point is not unique there

(any point with $|L(j\omega)| = 0\,\text{dB}$ & $\arg L(j\omega) = -180 \pmod{360}$ is critical).

Nyquist criterion on Nichols chart (contd)

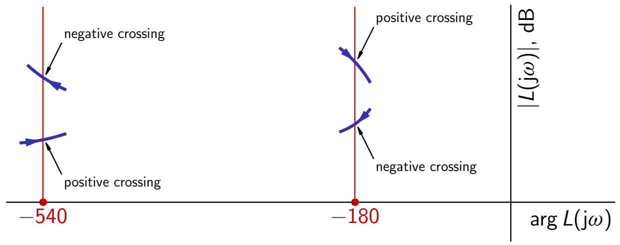
Remember (maybe; IC, Lect. 9):



Then the

- number of counterclockwise encirclements of $-1 + j0$ by the Nyquist plot of $L(j\omega)$ equals *twice* the net sum of crossings the ray $(-\infty, -1]$ by the polar plot of $L(j\omega)$ (plot direction is with the increase of ω).

The Nichols chart counterpart uses rays $[-180 + 360k, -180 + 360k + j\infty)$ for $k \in \mathbb{Z}$:



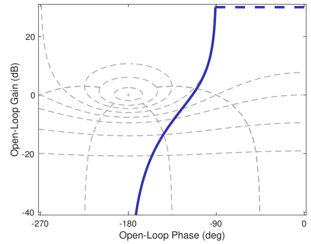
and the rest remains the same . . .

Nyquist criterion on Nichols chart: handling integrators

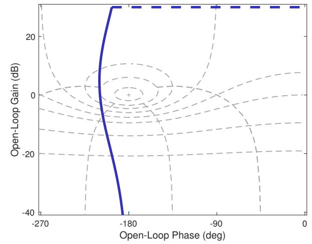
Polar plot: Each integrator adds a counterclockwise arc of 90° with infinite radius, starting at $L(j\omega)|_{\omega=0^+}$

Nichols chart: An arc centered at the origin has a constant magnitude and changing phase $\implies$ an arc translates to a horizontal line on Nichols chart:

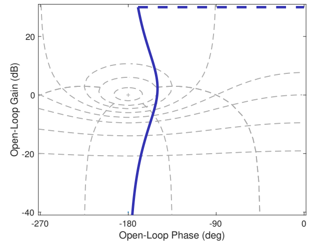
$L(s) = \dfrac{1}{2s(s+1)}$:



$L(s) = \dfrac{3(s+1)}{s^2(10s+3)}$:



$L(s) = \dfrac{3s+1}{4s^2(s+1)}$:

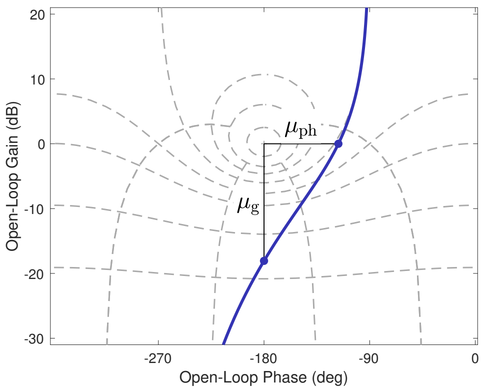


Each integrator needs −90° of the line.

Gain and phase margins on Nichols chart

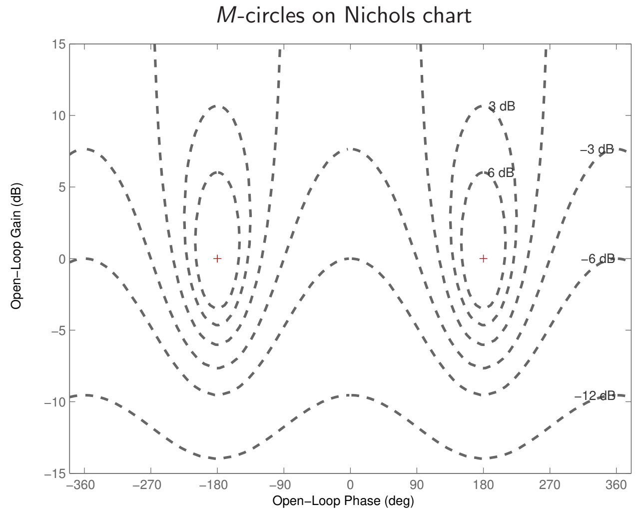
Gain margin μ_g and phase margin μ_{ph} are easily calculable from Nichols charts:

- μ_g is the vertical distance from the critical point;
- μ_{ph} is the horizontal distance from the critical point.

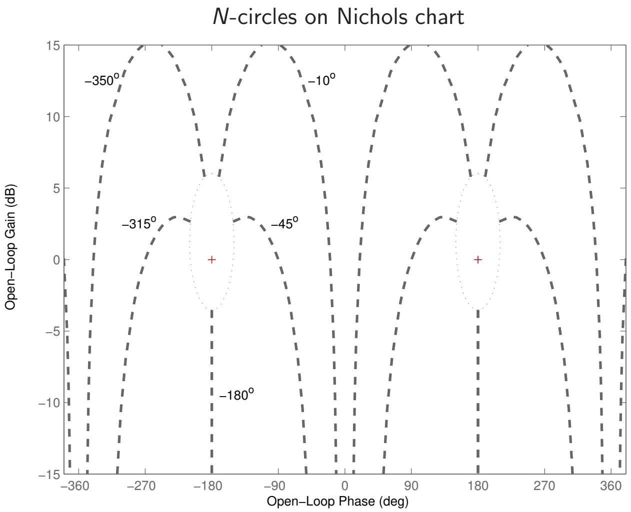


here $\mu_g = 8 \approx 18.06\,$dB and $\mu_{ph} \approx 63.36°$.

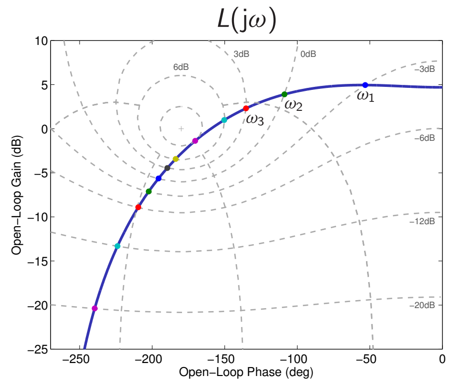
M-circles on Nichols charts

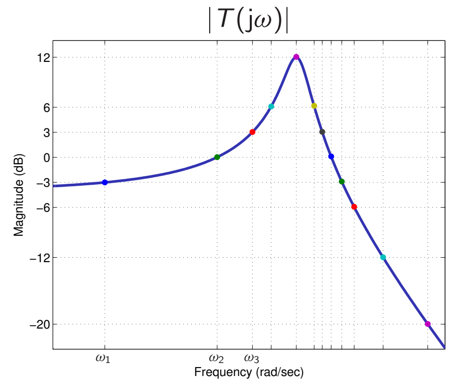


N-circles on Nichols charts



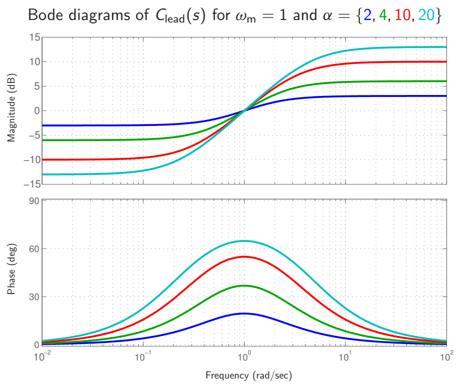
M-circles on Nichols charts: how to read

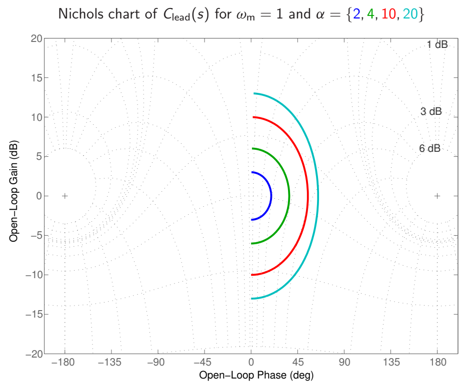




Nichols chart of lead controller

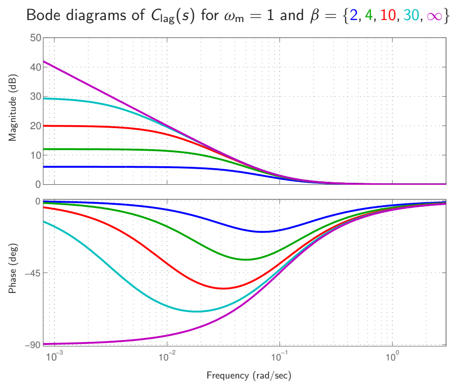
Lead controller: $C_{\text{lead}}(s) = \dfrac{\sqrt{\alpha}\, s + \omega_{\text{m}}}{s + \sqrt{\alpha}\omega_{\text{m}}}$, $\alpha > 1$.

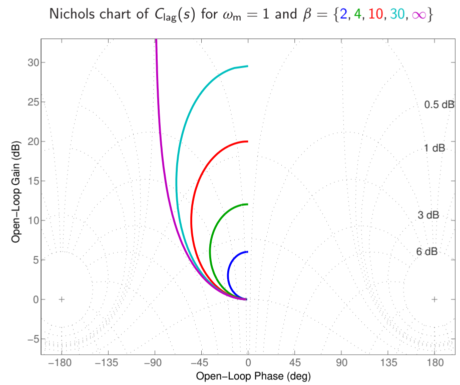




Nichols chart of lag controller

Lag controller: $C_{\text{lag}}(s) = \dfrac{10s + \omega_{\text{m}}}{10s + \omega_{\text{m}}/\beta}$, $\beta > 1$.





Outline

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Nichols chart

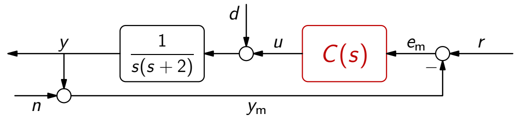
Design example: the use of Nichols charts

Bode's gain-phase relation

Philosophical remark: Bode's sensitivity integral

System (remember IC, Lect. 11)

A DC motor controlled in closed loop:



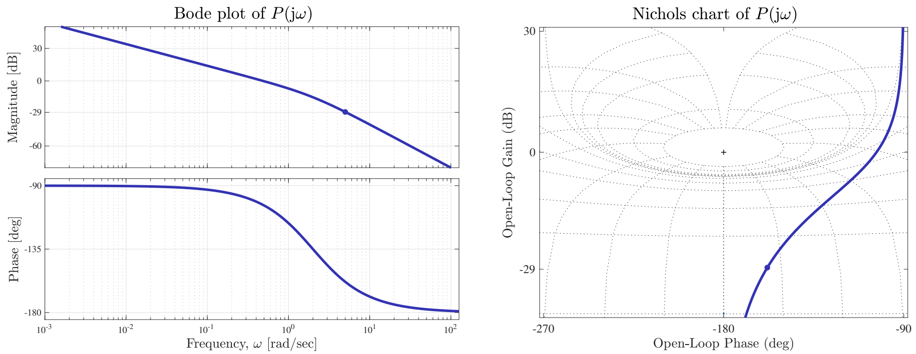
Requirements:

- closed-loop stability (of course)
- $\omega_c = 5$ rad/sec
- zero steady-state error for a step in r — always holds
- zero steady-state error for a step in d — integrator in $C(s)$
- $\mu_{ph} \in \{45, 60\}$

Remark: We implicitly assume that the plant is normalized, in a sense that the control amplitude $|u(t)| < 1$ is "small" and $|u(t)| > 1$ is "large".

Example 1: the plant

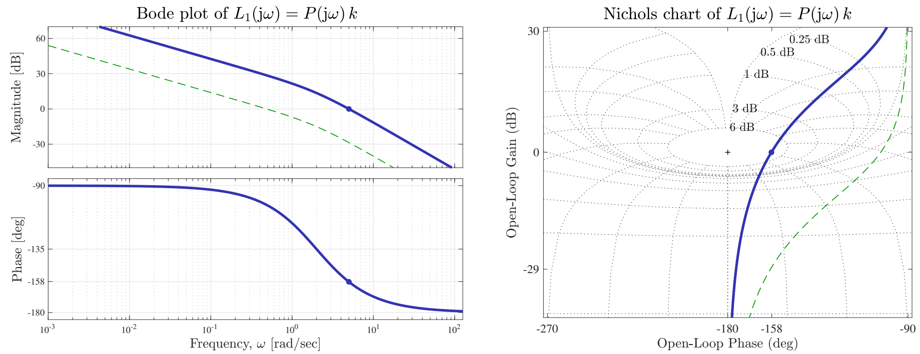
Let first $\mu_{ph} = 45°$. With $\omega_c = 5$,



This is below the actual crossover, so can be attained by the gain $k \approx 26.9$.

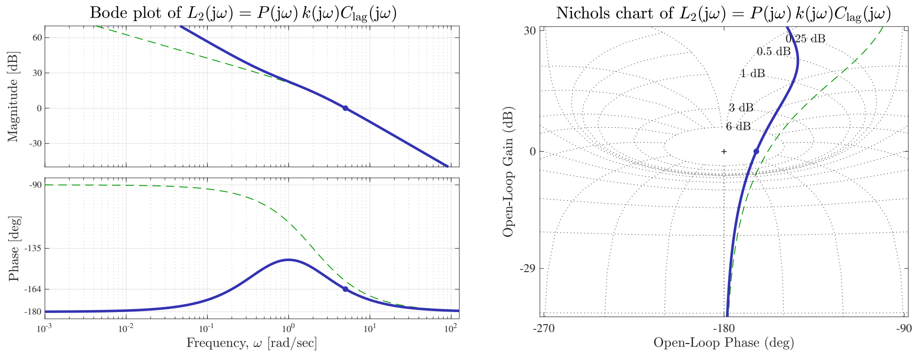
Example 1: adjusting crossover

We get:



Example 1: adjusting low-frequency gain

Use the lag controller with $\omega_m = 5$ and $\beta = \infty$:



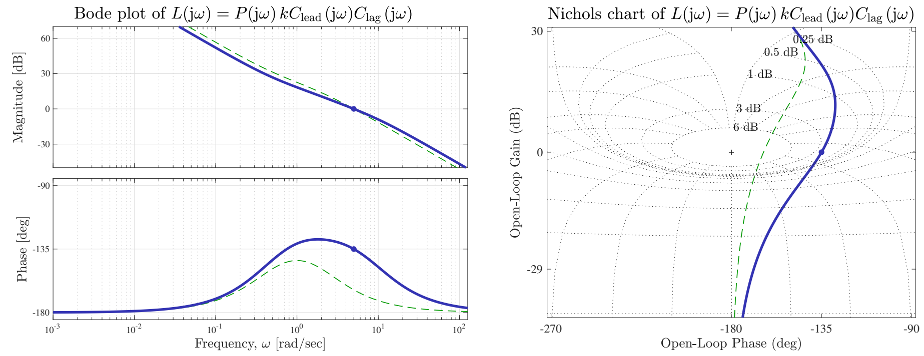
Here $\mu_{ph} \approx 16^\circ$ and

– we need a phase lead of $45^\circ - 16^\circ = 29^\circ$,

for which one lead is enough.

Example 1: adjusting phase around crossover

We get:

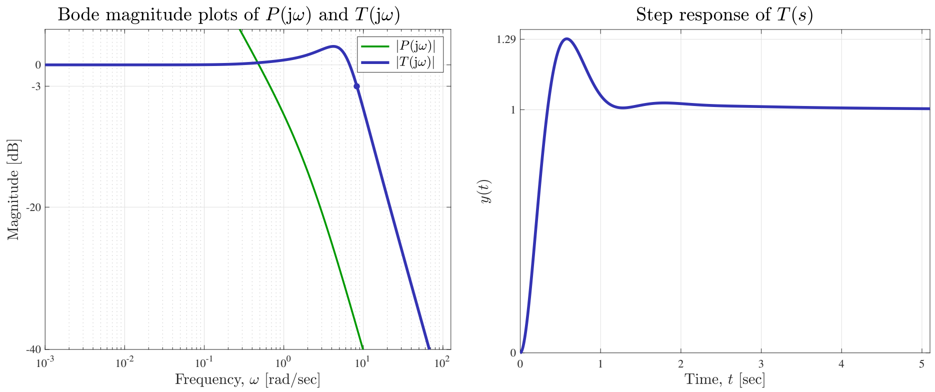


Here $\mu_{ph} \approx 45^\circ$, which is what we need. Resulting controller:

$$C(s) = kC_{lead}(s)C_{lag}(s) = \frac{45.619(s+2.951)(s+0.5)}{s(s+8.471)}.$$

Note Nichols chart location vis-à-vis M-circles (not quite "nice").

Example 1: closed-loop command response



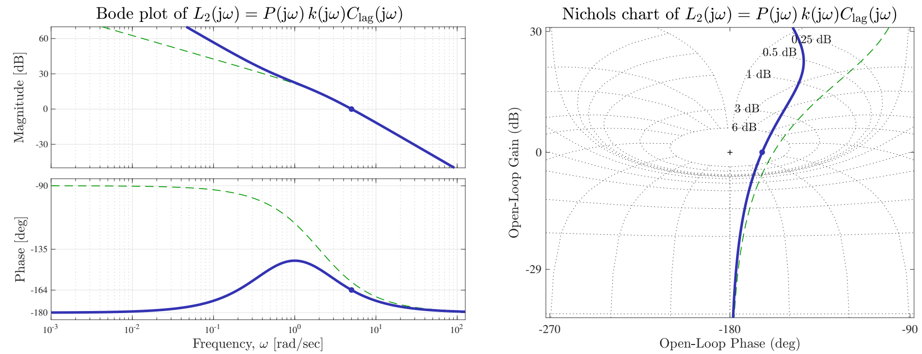
To note:

– resonance peak (agrees with M-circles) $\implies$ OS $\approx 29\%$

– closed-loop bandwidth $\omega_b \approx 8.3176$, which is a bit above the designed $\omega_c = 5$ and higher than the open-loop bandwidth

Example 2: adjusting low-frequency gain

Now, let $\mu_{ph} = 60^\circ$. The first design steps, up until the addition of the lag part, remain the same and we have



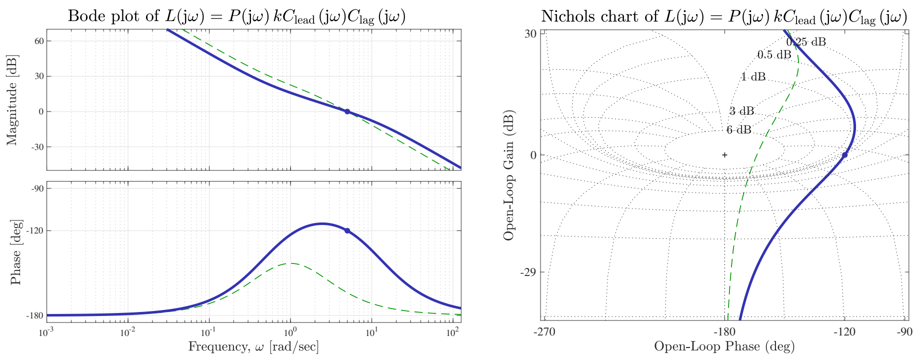
Here $\mu_{ph} \approx 16^\circ$ and

– we now need a phase lead of $60^\circ - 16^\circ = 44^\circ$,

for which one lead is enough as well.

Example 2: adjusting phase around crossover

We get:

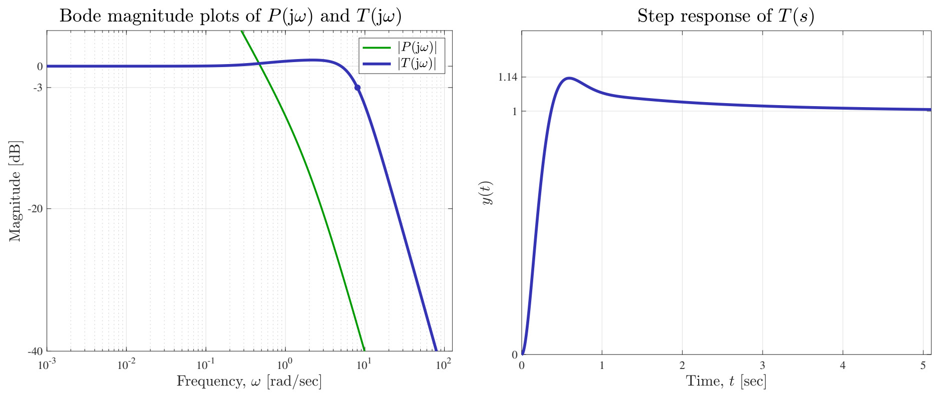


Here $\mu_{ph} \approx 60^\circ$, which is what we need. Resulting controller:

$$C(s) = kC_{lead}(s)C_{lag}(s) = \frac{62.977(s+2.127)(s+0.5)}{s(s+11.75)}.$$

Note again Nichols chart location vis-à-vis M-circles ("nicer").

Example 2: closed-loop command response

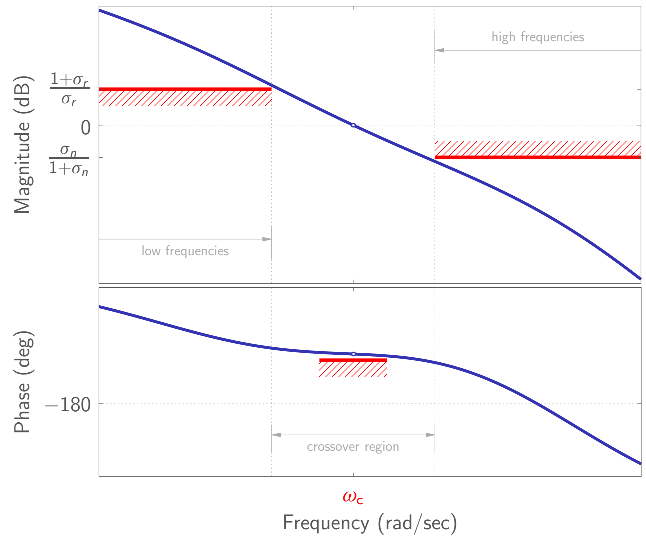


To note:

- resonance peak becomes lower $\implies$ lower OS $\approx 14\%$
- closed-loop bandwidth $\omega_b \approx 8.0649$, which is a bit above the designed $\omega_c = 5$ and higher than the open-loop bandwidth

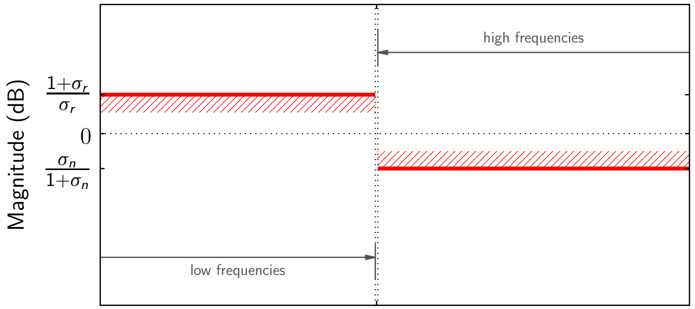
Outline

Loop shaping: big picture



Dream loop shape

We'd prefer to have narrow crossover region, something like this:



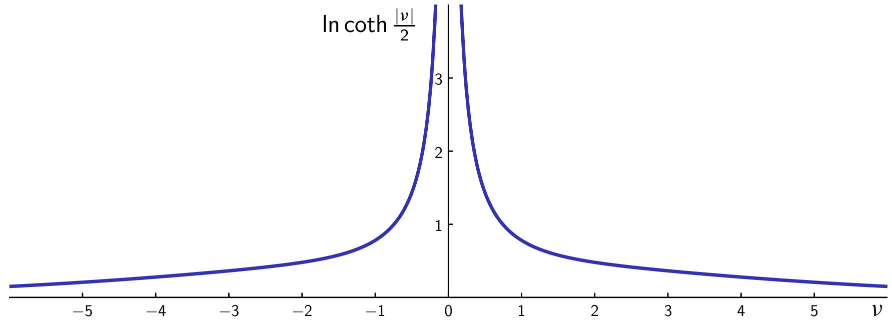
Intuitively, it is hard to believe that this is possible (too good to be true:-). It turns out that this "intuition" can be rigorously justified.

Bode's gain-phase relation: minimum-phase loop

Let $L(s)$ be stable and minimum-phase and such that $L(0) > 0$. Then $\forall \omega_0$

$$\arg L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d \ln|L(j\omega)|}{d\nu} \ln\coth\frac{|\nu|}{2} d\nu, \qquad \text{where } \nu := \ln\frac{\omega}{\omega_0}$$

($\coth x := \frac{e^x+e^{-x}}{e^x-e^{-x}}$). Function $\ln\coth\frac{|\nu|}{2} = \ln\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|$:



may be thought of as a rough approximation of the Dirac delta.

Bode's gain-phase relation: what does it mean

Since $\ln\coth\frac{|\nu|}{2}$ decreases rapidly as ω deviates from ω_0,

– $\arg L(j\omega_0)$ depends mostly on $\frac{d\ln|L(j\omega)|}{d\nu} = \frac{d\ln|L(j\omega)|}{d\ln\omega}$ near frequency ω_0.

But

– $\frac{d\ln|L(j\omega)|}{d\ln\omega} = \frac{d\log|L(j\omega)|}{d\log\omega}$ is the roll-off[2] of the Bode plot of $|L(j\omega)|$.

It can be shown that

$$\arg L(j\omega_0) < \begin{cases} -N\times 65.3^\circ, & \text{if roll-off of } |L(j\omega)| \text{ is } N \text{ for } \frac{1}{3} \le \frac{\omega}{\omega_0} \le 3 \\ -N\times 75.3^\circ, & \text{if roll-off of } |L(j\omega)| \text{ is } N \text{ for } \frac{1}{5} \le \frac{\omega}{\omega_0} \le 5 \\ -N\times 82.7^\circ, & \text{if roll-off of } |L(j\omega)| \text{ is } N \text{ for } \frac{1}{10} \le \frac{\omega}{\omega_0} \le 10 \end{cases}$$

In other words,

– high negative slope of $|L(j\omega)|$ necessarily causes large phase lag.

[2] Roll-off is the absolute value of the negative slope, scaled by 20.

Bode's gain-phase relation: implication

For systems with rigid loops, it is advisable to

– keep loop roll-off $\not\gg 1$ in the crossover region[3]

to guarantee that $L(j\omega)$ is far enough from the critical point. This, in turn, means that

– low- and high-frequency regions should be well-separated.

This is the reason why our "dream shape" is not an option.

[3] I.e. not much smaller than -20 dB/dec slope of $|L(j\omega)|$.

Gain-phase relation: one nonminimum-phase zero

Let $L(s)$ has one RHP zero at $z > 0$. Then

$$L(s) = \frac{-s+z}{s+z} L_{\mathrm{mp}}(s)$$

for a minimum-phase $L_{\mathrm{mp}}(s)$. Since $\left|\frac{-j\omega+z}{j\omega+z}\right| \equiv 1$, $|L(j\omega)| = |L_{\mathrm{mp}}(j\omega)|$ and

$$\begin{aligned}\arg L(j\omega_0) &= \arg L_{\mathrm{mp}}(j\omega_0) + \arg \frac{-j\omega_0 + z}{j\omega_0 + z} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d \ln|L(j\omega)|}{d\nu} \ln\coth\frac{|\nu|}{2} d\nu - 2\arctan\frac{\omega_0}{z}.\end{aligned}$$

Thus,

- nonminimum-phase zero adds a phase lag (especially at $\omega > z$)

imposing additional constraints on the slope of $|L(j\omega)|$ in crossover region.

Gain-phase relation: complex nonminimum-phase zeros

Now, let $L(s)$ has a pair of RHP zero at $z_r \pm jz_i$, $z_r > 0$. Then

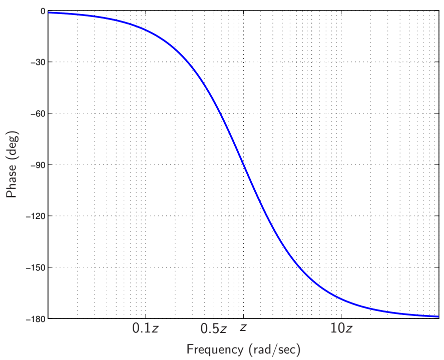
$$L(s) = \frac{-s+z_r+jz_i}{s+z_r+jz_i}\frac{-s+z_r-jz_i}{s+z_r-jz_i} L_{\mathrm{mp}}(s)$$

and we have:

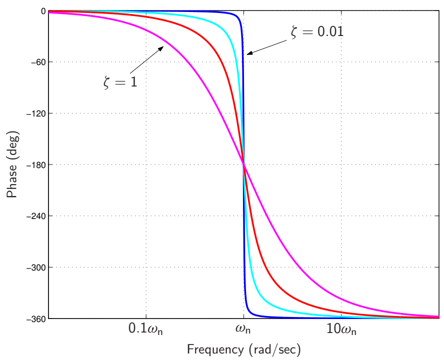
$$\begin{aligned}\arg L(j\omega_0) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d \ln|L(j\omega)|}{d\nu} \ln\coth\frac{|\nu|}{2} d\nu + \arg\frac{-j\omega_0 + z_r \pm jz_i}{j\omega_0 + z_r \pm jz_i} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d \ln|L(j\omega)|}{d\nu} \ln\coth\frac{|\nu|}{2} d\nu \\ &\quad - 2\Big(\arctan\frac{\omega_0 + z_i}{z_r} + \arctan\frac{\omega_0 - z_i}{z_r}\Big).\end{aligned}$$

This harden constraints when $\omega_0 > z_i$, though may soften when $\omega_0 \ll z_i$.

Phase of all-pass systems



$\arg \frac{-s+z}{s+z}\Big|_{s=j\omega}$



$\arg \frac{s^2 - 2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\Big|_{s=j\omega}$

Gain-phase relation: multiple nonminimum-phase zeros

In this case

$$L(s) = \frac{-s+z_1}{s+z_1}\frac{-s+z_2}{s+z_2}\cdots\frac{-s+z_k}{s+z_k} L_{\mathrm{mp}}(s)$$

and we have:

$$\arg L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d \ln|L(j\omega)|}{d\nu} \ln\coth\frac{|\nu|}{2} d\nu - \sum_{i=1}^{k} \arg\frac{-j\omega_0 + z_i}{j\omega_0 + z_i},$$

which further harden constraints.

Limitations due to nonminimum-phase zeros

For systems with a single crossover frequency[4] RHP zeros near ω_c

- impose additional limitations on the roll-off in the crossover region.

Consequently, a well-known rule of thumb says that for nonminimum-phase systems

- crossover frequency ω_c should be $<$ the smallest RHP zero.

Also, it is safe to claim (regarding RHP zeros) that

- closer to the real axis $\implies$ more restrictive crossover limitations

[4] For lightly damped systems it sometimes might be desirable to inject a phase lag by adding RHP zeros. Yet this must be done with maximal care (don't try it at home!)

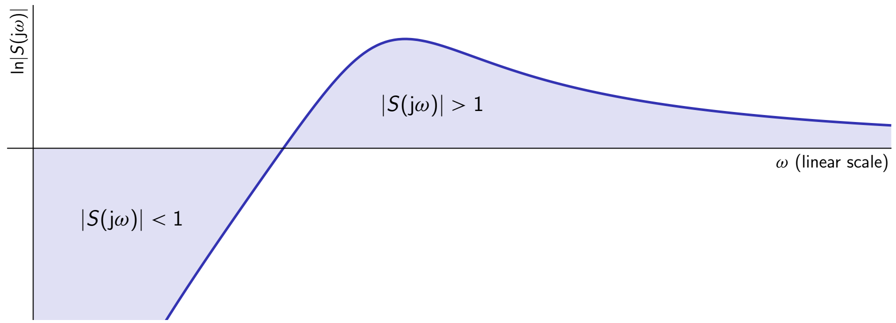
Outline

Bode's sensitivity integral

Let $L(s)$ be a loop transfer function having pole excess ≥ 2. Then, provided $S(s)$ is stable,

$$\int_0^\infty \ln|S(\mathrm{j}\omega)|\,\mathrm{d}\omega = \begin{cases} 0 & \text{if } L \text{ stable} \\ \pi \sum_{i=1}^m \operatorname{Re} p_i & \text{otherwise } (p_i\text{—unstable poles of } L) \end{cases}$$

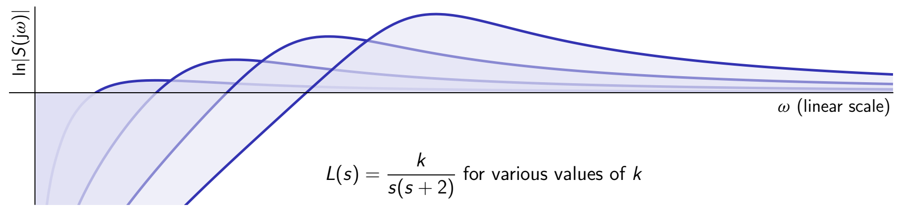
i.e.



What does it mean ?

Some conclusions:

- since $\pi \sum \operatorname{Re} p_i \geq 0$, $|S(\mathrm{j}\omega)|$ cannot[5] be < 1 over all frequencies
- improvements in one region inevitable cause deterioration in other (so-called waterbed effect)



On qualitative level,

- controller can only redistribute $|S(\mathrm{j}\omega)|$ over frequencies

and the art of control may thus be seen as art of redistribution of $|S(\mathrm{j}\omega)|$.

[5] If pole excess of $L(s)$ is ≥ 2, of course. Yet this is typical in applications.