תורת הבקרה (035188)

גליון תרגילי בית מס' 2

שאלה מס' 1

מערכת נתונה ע"י פונקצית התמסורת $P(s) = \frac{s+a}{s^2}$ מהכניסה u ליציאה y.

1. כתבו למערכת מימוש כרצונכם במרחב המצב, מהצורה

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \\ y(t) = Cx(t). \end{cases}$$

2. בדקו קונטרולביליות של (A, B). באם קיימים ערכי a עבורם המערכת אינה קונטרולבילית, חשבו את המודים שאינם קונטרולביליים.
3. בדקו אובזרוביליות של (C, A). באם קיימים ערכי a עבורם המערכת אינה אובזרובילית, חשבו את המודים שאינם אובזרוביליים.
4. תכננו משוב מצב מהצורה $u(t) = r(t) - Fx(t)$ לייצוב המערכת, כך שפונקצית התמסורת מ־r אל y תהיה מסדר ראשון. מהם ערכי a עבורם זה אפשרי?
5. מהו הסדר המינימלי של פונקצית התמסורת מ־r אל y שניתן להשיג ע"י משערך + משוב מצב?
6. מהם ערכי a עבורם בעיית ה־LQR עם פונקצית המחיר

$$\mathcal{J}_c = \int_0^\infty \big(\lambda y(t)^2 + (1-\lambda)\dot{y}(t)^2 + u(t)^2\big)dt, \tag{1}$$

כאשר $0 < \lambda \leqslant 1$ ניתנת לפתרון? חשבו את הפתרון.

שאלה מס' 2

נתונה המערכת $P(s) = \frac{1}{s^2}$ ופונקצית המחיר (1). מצאו את ערך $0 < \lambda \leqslant 1$ עבורו לחוג הסגור עם בקר LQR אופטימלי יש קוטב כפול (ללא פתרון של משוואת ריקאטי).

1. Companion form:
$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} a & 1 \end{bmatrix} x(t). \end{cases}$$
Note that one may also choose any other realization, like the observer form. In this case items 2, 3, and 6 will be different.

2. Companion form is always controllable. This can be seen, for example, from the controllability matrix $M_c = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

3. As the second-order realization above is minimal iff it is both controllable and observable and as it is always controllable, this realization can be unobservable iff there are pole/zero cancellations in $P(s)$. This is obviously the case only when $a = 0$, in which case one pole at the origin is unobservable. An alternative way to see this is through the observability matrix $M_o = \left[\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right]$, which is singular only for $a = 0$. As we have modes only at the origin, the unobservable mode is obviously the one at $s = 0$.

4. We know (Lecture 8, p. 4) that zeros of the closed-loop transfer function from r to y under a state-feedback control law are those of the plant. Thus, in order to end up with a first-order transfer function we have to guarantee that one closed-loop pole cancels the plant zero at $s = -a$. This cancellation is possible iff $a > 0$ (closed-loop poles must be stable as feedback is stabilizing). Thus, we have to assign the following closed-loop characteristic polynomial:
$$\chi_{cl}(s) = (s + \lambda)(s + a) = s^2 + (a + \lambda)s + a\lambda$$
for any $\lambda > 0$. The rest is trivial (remember, the realization is in the companion form):
$$F = \begin{bmatrix} a\lambda & a + \lambda \end{bmatrix}.$$

5. We know (Lecture 8, p. 16) that the closed-loop transfer function from r to y in the observer-based configuration does not depend on the observer and is the same as in the state-feedback case. Thus, it should in general be a second-order transfer function with one zero at $s = -a$. Following the reasoning in the previous item, if $a > 0$, the order can be reduced by putting one of the closed-loop poles to $-a$. In this case the minimal order is 1. On the other hand, if $a \leqslant 0$, this zero cannot be canceled and the minimal order is 2.

6. Note that $y(t) = \begin{bmatrix} a & 1 \end{bmatrix} x(t)$ and $\dot{y}(t) = \begin{bmatrix} 0 & a \end{bmatrix} x(t) + u(t)$. Hence,
$$\lambda y^2 + (1-\lambda)\dot{y}^2 + u^2 = x' \underbrace{\left(\lambda \begin{bmatrix} a \\ 1 \end{bmatrix} \begin{bmatrix} a & 1 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 0 \\ a \end{bmatrix} \begin{bmatrix} 0 & a \end{bmatrix} \right)}_{C_z' C_z} x + 2x' \underbrace{(1-\lambda) \begin{bmatrix} 0 \\ a \end{bmatrix}}_{S} u + \underbrace{(2-\lambda)}_{\rho} u^2$$
We know (Lecture 9, p. 16) that cost function results in the LQR problem with the plant
$$\dot{x}(t) = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \frac{1-\lambda}{2-\lambda} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & a \end{bmatrix} \right) x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{u}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{1-\lambda}{2-\lambda} a \end{bmatrix}}_{\tilde{A}} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{u}(t), \tag{2}$$
where $u(t) = \tilde{u}(t) - \frac{1-\lambda}{2-\lambda} \begin{bmatrix} 0 & a \end{bmatrix} x(t)$, and the cost function with $\rho = 2 - \lambda$ and any $\tilde{C}_z$ such that
$$\tilde{C}_z' \tilde{C}_z = \lambda \underbrace{\begin{bmatrix} a \\ 1 \end{bmatrix} \begin{bmatrix} a & 1 \end{bmatrix}}_{C'C} + (1-\lambda) \underbrace{\begin{bmatrix} 0 \\ a \end{bmatrix} \begin{bmatrix} 0 & a \end{bmatrix}}_{A'C'CA} - \underbrace{\frac{(1-\lambda)^2}{2-\lambda} \begin{bmatrix} 0 \\ a \end{bmatrix} \begin{bmatrix} 0 & a \end{bmatrix}}_{\frac{1}{\rho} SS'} = \begin{bmatrix} \lambda a^2 & \lambda a \\ \lambda a & \lambda + \frac{1-\lambda}{2-\lambda} a^2 \end{bmatrix}.$$
One possible choice is
$$\tilde{C}_z = \begin{bmatrix} \sqrt{\lambda} a & \sqrt{\lambda} \\ 0 & \sqrt{\frac{1-\lambda}{2-\lambda}}\, a \end{bmatrix}.$$

Now, the LQR problem for this data is solvable iff realization (2) is stabilizable (always true as it is in the companion form, hence always controllable) and $(\tilde{C}_z, \tilde{A})$ has no unobservable modes on the $j\omega$-axis. To check this condition, note that $\tilde{A}$ has two modes at $s = 0$ and $s = -\frac{1-\lambda}{2-\lambda}\,a$. To see, whether they are observable, let's use the PBH test (Lecture 7, p. 11). If the first mode is not observable, then the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & -\frac{1-\lambda}{2-\lambda}\,a \\ \sqrt{\lambda}a & \sqrt{\lambda} \\ 0 & \sqrt{\frac{1-\lambda}{2-\lambda}}\,a \end{bmatrix}$$

has reduced column rank. This is obviously true iff $a = 0$ and in this case the problem is not solvable. If the mode at $-\frac{1-\lambda}{2-\lambda}\,a$ is unobservable, the matrix

$$\begin{bmatrix} \frac{1-\lambda}{2-\lambda}\,a & 1 \\ 0 & 0 \\ \sqrt{\lambda}a & \sqrt{\lambda} \\ 0 & \sqrt{\frac{1-\lambda}{2-\lambda}}\,a \end{bmatrix}$$

has reduced column rank. This is again true iff $a = 0$ (the columns are linearly dependent). To conclude, the problem is solvable iff $a \neq 0$.

Assume now that $a \neq 0$. Then the optimal control law is

$$u(t) = -\frac{1}{2-\lambda}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} x(t) - \frac{1-\lambda}{2-\lambda}\begin{bmatrix} 0 & a \end{bmatrix} x(t) = -\frac{1}{2-\lambda}\begin{bmatrix} p_2 & p_3 + (1-\lambda)a \end{bmatrix} x(t),$$

where $\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$ is the stabillizing solution of the Riccati equation

$$\begin{bmatrix} 0 & 0 \\ 1 & -\frac{1-\lambda}{2-\lambda}\,a \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -\frac{1-\lambda}{2-\lambda}\,a \end{bmatrix} + \begin{bmatrix} \lambda a^2 & \lambda a \\ \lambda a & \lambda + \frac{1-\lambda}{2-\lambda}\,a^2 \end{bmatrix} - \frac{1}{2-\lambda}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = 0.$$

Since we are looking for the stabilizing solution, the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & -\frac{1-\lambda}{2-\lambda}\,a \end{bmatrix} - \frac{1}{2-\lambda}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2-\lambda}\,p_2 & -\frac{1}{2-\lambda}((1-\lambda)a + p_3) \end{bmatrix}$$

must be stable. In other words, the conditions $p_2 > 0$ and $p_3 > (\lambda - 1)a$ must hold.

The ARE can be rewritten as

$$\begin{bmatrix} \lambda a^2 - \frac{1}{2-\lambda}\,p_2^2 & p_1 - \frac{1-\lambda}{2-\lambda}\,ap_2 + \lambda a - \frac{1}{2-\lambda}\,p_2p_3 \\ p_1 - \frac{1-\lambda}{2-\lambda}\,ap_2 + \lambda a - \frac{1}{2-\lambda}\,p_2p_3 & 2(p_2 - \frac{1-\lambda}{2-\lambda}\,ap_3) + \lambda + \frac{1-\lambda}{2-\lambda}\,a^2 - \frac{1}{2-\lambda}\,p_3^2 \end{bmatrix} = 0.$$

The $(1,1)$ element yields

$$p_2 = \sqrt{\lambda(2-\lambda)}\,|a|$$

(remember that $p_2 > 0$). The $(2,2)$ element yields

$$p_3^2 + 2a(1-\lambda)p_3 - \big((2-\lambda)(2p_2 + \lambda) + (1-\lambda)a^2\big) = 0$$

Therefore,

$$p_3 = -a(1-\lambda) \pm \sqrt{a^2(1-\lambda)^2 + (2-\lambda)(2p_2 + \lambda) + (1-\lambda)a^2}.$$

Since $p_3 > -(1-\lambda)a$ and $(2-\lambda)(2p_2 + \lambda) + (1-\lambda)a^2 > 0$, we have that

$$p_3 = -a(1-\lambda) + \sqrt{(2-\lambda)\big(a^2(1-\lambda) + 2\sqrt{\lambda(2-\lambda)}\,|a| + \lambda\big)}.$$

The last parameter, p_1, can be easily obtained from the $(1,2)$ element. Yet since the optimal control doesn't depend on p_1, this step is not required. Thus, the control law is

$$u(t) = -\frac{1}{2-\lambda}\begin{bmatrix} \sqrt{\lambda(2-\lambda)}\,|a| & \sqrt{(2-\lambda)\big(a^2(1-\lambda) + 2\sqrt{\lambda(2-\lambda)}\,|a| + \lambda\big)} \end{bmatrix} x(t).$$

Hmm, it looks that I slightly exaggerated with this item...

פתרון לשאלה מס' 2

In this case, the problem is the LQR for the plant

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

and the regulated signal

$$z(t) = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{1-\lambda} \end{bmatrix} x(t)$$

(since $x_1 = y$ and $x_2 = \dot{y}$). Hence, we have that $C_z = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{1-\lambda} \end{bmatrix}$ and $\rho = 1$.

It follows from the return-difference equality (Lecture 9, p. 11) that

$$\chi_{\mathrm{ol}}(-s)\chi_{\mathrm{ol}}(s) + \phi(s) = \chi_{\mathrm{cl}}(-s)\chi_{\mathrm{cl}}(s),$$

where $\phi(s)$ is the numerator polynomial of

$$\begin{aligned} \tfrac{1}{\rho} P_z(-s)' P_z(s) &= B'(-sI - A')^{-1} C_z' C_z (sI - A)^{-1} B = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -s & 0 \\ -1 & -s \end{bmatrix}^{-1} \begin{bmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{s^2} & -\frac{1}{s} \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} \frac{1}{s^2} \\ \frac{1}{s} \end{bmatrix} = \frac{\lambda}{s^4} - \frac{1-\lambda}{s^2} = \frac{\lambda - (1-\lambda)s^2}{s^4}. \end{aligned}$$

Thus, we have that

$$\chi_{\mathrm{cl}}(-s)\chi_{\mathrm{cl}}(s) = s^4 - (1-\lambda)s^2 + \lambda.$$

Therefore, to get a double closed-loop pole at $-\alpha$ for some $\alpha > 0$ we have to meet the following requirement:

$$s^4 - (1-\lambda)s^2 + \lambda = (-s+\alpha)^2(s+\alpha)^2 = (\alpha^2 - s^2)^2 = s^4 - 2\alpha^2 s^2 + \alpha^4.$$

This leads to $\lambda = \alpha^4$ and then

$$1 - \alpha^4 = 2\alpha^2 \iff \alpha^4 + 2\alpha^2 - 1 = 0 \iff \alpha^2 = -1 + \sqrt{2}.$$

Thus, the required

$$\lambda = (\sqrt{2} - 1)^2 = 3 - 2\sqrt{2} \approx 0.17157.$$